ISLAMIC PATTERNS AND SYMMETRY GROUPS

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Introduction

There have been many attempts to describe what mathematics is, and in the classic book by Courant and Robbins (1941/1996) the authors intend, through examples from many areas of mathematics, to give an impression of this. They state that they do not intend to go into a detailed philosophical analysis of mathematics but rather they want to paint a picture as broad as possible by visiting many fields without going into great details in any of these fields. However, they warn against placing too much emphasis on the deductive-postulational character of mathematics. Later Hersh (1997), who in the title of his book is alluding to the book by Courant and Robbins, takes the stance that "mathematics must be understood as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context" (p. xi). Devlin (1994) tries to define mathematics by looking at some common features characterising the activities that mathematicians are occupied with. His conclusion, which he claims to be commonly accepted, is that "mathematics is *the science of patterns*" (p. 3, emphasis in original). He goes on to say that the patterns that mathematicians study could be numerical patterns, patterns of shape, patterns of motion, to name some examples. G.H. Hardy, in his famous book from 1940 (Hardy, 1940/1992, p. 84), writes that the patterns of mathematics are made with *ideas*. He contrasts the mathematician's patterns with the patterns of a painter which are made with shapes and colours, and those of a poet which are made with words. He recognises that also painters and poets, to some extent, work with ideas but "[a] mathematician, on the other hand, has no material to work with but ideas" (pp. 84-85). It is exactly at this place that Hardy writes what probably is the most famous statement from this book.

> The mathematician' patterns, like the painter's or the poet's must be *beautiful*; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics. (Hardy, 1940/1992, p. 85, emphasis in original)

To distinguish between what is 'beautiful' or 'ugly' within mathematics will probably be based on culturally situated criteria, in the same way as a similar distinction in art will be. Hardy's analogies between art and mathematics may give rise to the question whether the research mathematician's activities can be likened to artistic creativity. Michele Emmer has further linked this question to a fundamental question in the philosophy of mathematics, namely whether mathematics is *invented* or *discovered* (Emmer, 2005).

Rephrasing Hardy's word about the mathematician as someone who works with ideas into more modern language from mathematics education one could for example characterise the numerical patterns or the patterns of shape that the mathematician works with as signs or symbol representing some abstract mathematical concept. When these patterns are used for decorative purposes, this could be regarded as an object or a reference context for the mathematical concept (e.g. Steinbring, 2006). The mathematical concepts are *ideal* in the

sense that no reference context can fully capture exactly the properties of the concept. A sign representing a mathematical concept can be of different types. If it is an *iconic* sign, representing the concept by likeness (Peirce, 1998, p. 13), it will never have exactly the properties of the concept for which it stands. A *symbolic* sign, however, represents the concept by convention, and therefore its meaning is given to be exactly that of the concept for which it stands.

In this article I will use particular patterns of shape taken from Islamic culture as a reference context. These patterns can be connected to certain mathematical concepts, and the patterns themselves can be viewed as iconic signs of these mathematical concepts. I will also present other sign systems for representing them, and discuss certain issues that will occur when trying to make these connections. In my description of the various patterns I will sometimes use statements like 'this pattern is a *p2*" when I really mean that the pattern has properties which are characteristic of the group represented by the sign *p2*. I will come back to this particular notation later. I choose this rather imprecise way of talking about the patterns because it would be too cumbersome to write in a more precise way every time.

Geometry is a wide and rich branch of mathematics and it makes sense to divide geometry into subtopics. Henderson and Taimina (2005, pp. 1-8) describe four strands along which geometry has developed. They give the following names to these four strands.

1. Art/Pattern
2. Navigation/Stargazing
3. Building Structures
4. Motion/Machines

The Art/Pattern strand is very much concerned with studying patterns and symmetry and is not so much occupied with measurements and calculations. These topics are however very central in the Navigation/Stargazing strand where problems connected to exact positioning have been fundamental. Although the Art/Pattern strand will be the most conspicuous in discussing Islamic patterns I will later argue that the patterns could be inspired by work in the Navigation/Stargazing strand.

Geometry is a part of mathematics which exists in a tension between abstract theory and concrete practice. The four strands listed above all take as their starting point a practical approach. On the other hand the theoretical (axiomatic, deductive) approach has been very strong since the time of Euclid. This tension between geometry as an axiomatic, logical deductive structure and the description of the space surrounding us, and its implications for education has been discussed by Freudenthal (1973). It is also pointed out by Hershkowitz (1990) who discusses the ontological status of the geometrical entities. "The question is whether the geometrical entities are part pf the physical world, and if not, what are they?" (Hershkowitz, p. 92). In his work on intuition in mathematics Fischbein takes visualisation to be the main factor contributing to intuitive knowledge. "Its role is so important that very often intuitive knowledge is identified with visual representation", he writes (Fischbein, 1987, p. 103). In a later paper he elaborates on the nature of geometrical concepts in particular. He assigns to them a dual nature. On the one hand, he writes, "[t]he properties of geometrical figures are imposed by, or derived from definitions in the realm of a certain axiomatic system. From this point of view, also, a geometrical figure has a conceptual nature." On the other hand, "a geometrical figure is not a mere concept. It is an image, a visual image. It possesses a property which usual concepts do not possess, namely, it includes

the mental representation of space property" (Fischbein, 1993, p. 141). This is the background for referring to geometrical concepts as *figural concepts.*

1. Islamic patterns

It is well known that in Islam it is strictly forbidden to depict the deity. This does not mean however that Islamic art is entirely non-figurative, and of course geometric patterns used for decorations are abundant also in other religions and cultures. So when geometric patterns are so closely connected to Islamic art this may have to do with the fact that buildings, religious and others, in the Muslim world are so overwhelmingly richly decorated. Often there is hardly a surface which is not filled with decorative patterns of some sort. The patterns are frequently space filling, meaning that they are made from one basic cell which is repeated in two directions. Such patterns are often referred to as *tilings* or *tessellations.* In the tiles one often finds quite complex polygons that can be constructed using a number of circles. Furthermore these polygons often have the shape of a star with six or eight points, or multiples thereof.

A basic shape, frequently observed, but sometimes distorted so that it is hardly recognisable, is based on two squares inscribed in a circle so as to form an eight pointed star. One square is rotated 45 degrees with respect to the other. The shape is shown in Figure 1, and it is referred to as *Khatem Sulemani* meaning *Solomon's seal* (Abas & Salman, 1995). This shape itself is not space filling. If it is repeated horizontally and vertically a cross shaped region will be formed between four of the stars as shown in Figure 2. Together the star and the cross will cover the plane.
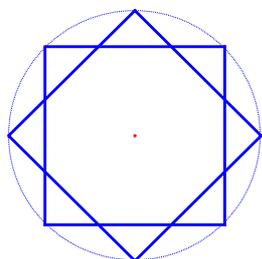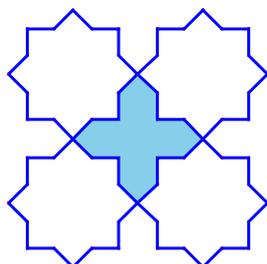

Figure 1

Figure 2

The Khatem Sulemani becomes more interesting when it is elaborated on. Starting with the basic pattern in Figure 1, make two smaller circles with the same centre as the original circle. On each of the new circles place eight points, equally spaced. Use these points to make an eight pointed star radiating from the innermost circle. The details of the construction are shown in Figure 3. The radii of the smaller circles are arbitrary, and when they are varied the star takes on new interesting shapes.
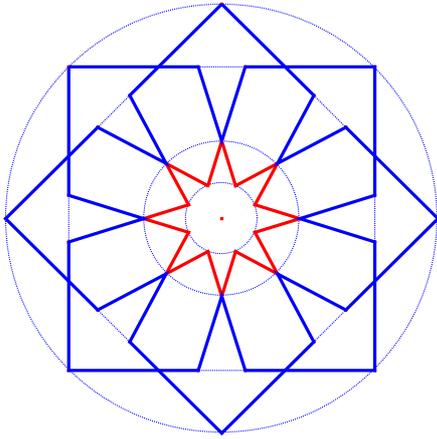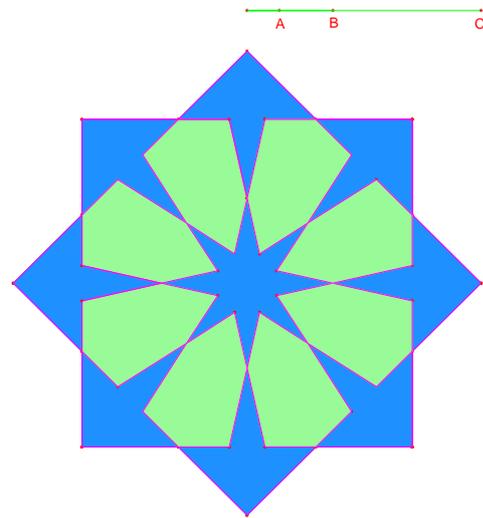
Figure 3

Figure 4

Figure 4 shows the star with different parts in different colours. To see the effect of varying the radius of the inner circles, click on the picture in Figure 4. This opens a dynamic web page with the picture. By dragging the points A and B on the segment above the picture, the radii of the inner and middle circle will change. Figure 5[1], which is a picture from Taj Mahal in India, shows how the Khatem Sulemani is the basis for a frieze pattern.
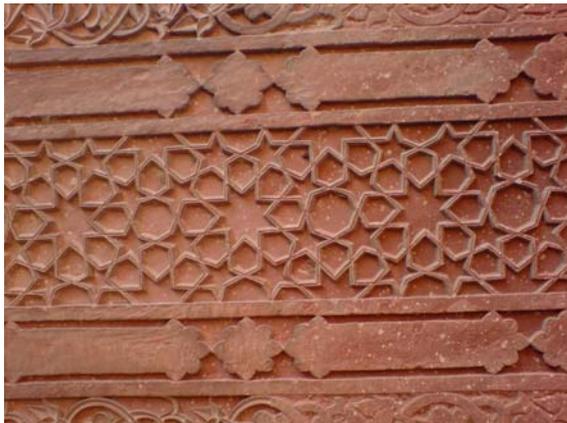

Figure 5

The star in Figure 4 is often used as the basis for patterns characterised by stars having eight, or a multiple of eight, points. However, also stars with six, or multiplies of six, points are frequently found in Islamic patterns. Such stars can also be obtained as a variation of Khatem Sulemani. Imagine starting with one square in Figure 1, and instead of rotating 45 degrees rotate 30 and another 30 degrees. This gives a 12 pointed star, shown in Figure 6. Removing the 12 points touching the outermost circle and colouring what is left, a star pattern very often seen in Islamic patterns emerges. This is shown in Figure 7. In the same way as with Figure 4, Figure 7 becomes interactive when clicking on it. Dragging the points A and B changes the radii of the inner circles, and the shape of the star can be made to change dramatically.

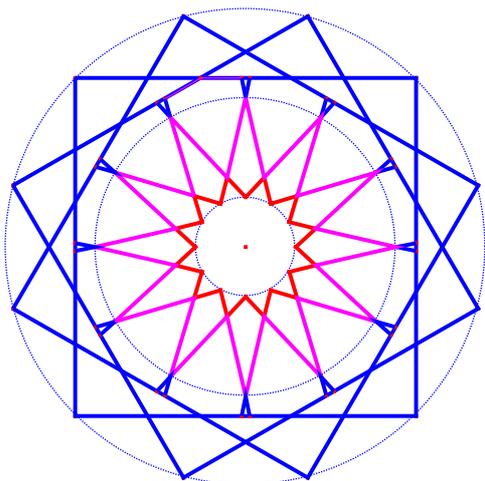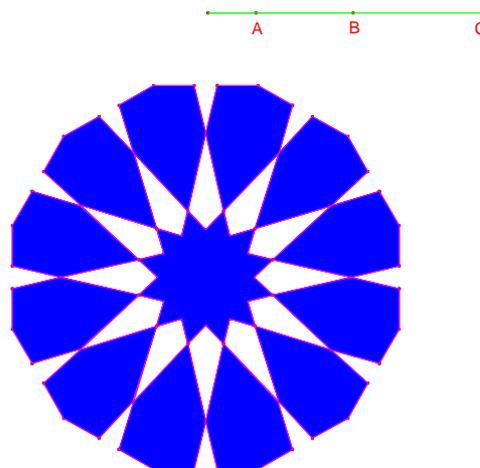[1] All photos in the article are taken by the author.

Figure 6

Figure 7

The figures are made using Cabri II plus, version 1.4.3. To manipulate the figures it is necessary to download Cabri II Plus 1.4 Plugin and to allow for active content when loading the figure. The plugin can be obtained for free from www.cabri.com. Details of the constructions can be found in (Abas & Salman, 1995).

The fact that stars are so abundant in Islamic patterns may form a link between the Art/Pattern strand and the Navigation/Stargaizing strand. As Abas and Salman (1995) point out astronomic observations were made very early in Muslim cultures, and indeed the authors claim (p. 11) that the first astronomical observatory where extensive observations and computations were made was set up in Maraghah in Iran in the 13th century. The interest for astronomic observation can be explained from a religious point of view, since at any time a Muslim needs to know the direction towards Mecca in order to perform the daily prayers. However, the need for precise astronomical information may also be rooted in more profane needs. Many Muslim cultures were nomadic tribes living in the desert where obviously navigating by the stars was an important skill. There were also great sea farers among the Arabs for whom this skill was equally important. Geometric constructions in connection with astronomy are to a large extent based on combining circles in a clever way, and this would fit well with the observation that many Islamic patterns are based on constructions using circles.

So far I have avoided to give a definition of what should be meant by an Islamic pattern. There is of course no such commonly accepted definition, and it might also be argued that it suffices to have an intuitive perception of what an Islamic pattern is. To show that this question has been addressed I will provide the following definition.

An Islamic pattern is one which satisfies one or more of the following criteria:

1. The pattern is transcribed with Arabic Calligraphy from the Quran.

2. The pattern was invented between 900 A.D. and 1500 A.D. and was used to decorate architectural surfaces or other works of art for Muslims, in a culture where the majority of the population, or at least the ruling element, professed the faith of Islam.

3. The pattern is derived from one or more patterns which satisfy criterion 2 and is such that the characteristic shapes from the original (or originals) are recognizable.
(Abas & Salman, 1995, p. 7)

In the discussions that will be presented later the question of what should be counted as an Islamic pattern will be of some importance.

For further examples of Islamic patterns and descriptions of how they can be constructed, see for example (Broug, 2008).

2. Symmetry groups

Geometric patterns are often investigated based on their symmetry properties. This gives a strong link between geometry and algebra since the collection of symmetry operations on a figure or a pattern constitutes a *group*. Islamic patterns are often repeating in two non-parallel directions, and they often cover large areas (walls and ceilings). Therefore many of them can easily be imagined as continuing indefinitely, and hence they have translational symmetries in non-parallell directions. In addition they may or may not contain other symmetries; rotations, reflections or glide-reflections.

Every pattern with translational symmetry in two non-parallel directions can be characterised in terms of its symmetry group. It is a remarkable fact that when investigating all possibilities it can be shown that there exist, up to isomorphism, exactly 17 such groups (patterns). These are commonly known as the *wallpaper groups*. A table and a description of the groups, as well as a proof of why there are exactly 17, can for example be found in (Martin, 1982). There are also numerous illustrations and descriptions available on the Internet. See for example http://mathworld.wolfram.com/WallpaperGroups.html (Weisstein, n.d.). On http://www.clowder.net/hop/17walppr/17walppr.html (David, n.d.) I found an animated table where it is possible to see the symmetry transformations contained in each pattern. It seems worth mentioning that the number of possibilities, 17, is a prime number. An intuitive thought might be that the number of possibilities would emerge from a combinatorial argument but that would rule out the possibility of a prime number.

In a pattern with no translations, like in a bounded figure, any rotational symmetry is possible. For example a regular n-gon will have rotational symmetry $360/n$ degrees. In addition the n-gon will have a number of reflections. The symmetries of the regular n-gon can be described by the *dihedral group, D_n*. This is a non-commutative group with $2n$ elements. To the contrary, in a wallpaper pattern the possibilities for rotational symmetries are very restricted. The only possible rotations are 60, 90, 120, and 180 degrees, often referred to as rotations of order 6, 4, 3, and 2, respectively. This severe restriction on possible rotations is often referred to as the *Crystallographic Restriction*, and for details I refer to (Martin, 1982). Since the symmetries form a group it is clear that certain combinations of rotations are impossible. For example a pattern cannot simultaneously contain rotations of order 3 or 6 and 4 (120 or 60 and 90 degrees) since that would imply that it would, by a suitable combination, have to contain a rotation of 30 degrees, which is impossible by the Crystallographic Restriction. The interest in symmetries of lattices is naturally strong among people working with crystals. They would first and foremost be interested in symmetries in three dimensions but also two dimensional symmetries would be relevant. The first classification of the 17 wallpaper groups is usually credited to the crystallographer Federov who published the table in a journal on mineralogy in 1891 (Coxeter, 1969, p. 50). The

notation commonly used for the groups also comes from crystallography. Each group is denoted by a symbol of the form *pnxy* or *cnxy*, where *p* or *c* is used depending on the unit cell of the lattice, *n* is a number denoting the order of rotational symmetry, and *x* and *y* are letters indicating the type of reflections or glide-reflections present. If there can be no confusion some parts of the symbol may be omitted. For further details about the notation, see (Horne, 2000, pp. 31-37) where also a flow chart that can be used to determine the group of a given pattern can be found. A similar flow chart can also be found in (Abas & Salman, 1995, pp. 109-113). I have chosen not to present a full table of all the 17 patterns here. The reader is advised to use one of the web addresses given above, or other readily available sources, as references when a particular pattern in mentioned.

3. Islamic patterns and wallpaper groups

Any Islamic pattern, seen as stretching and repeating indefinitely in two directions, can of course be classified to belong to one of the 17 wallpaper groups. If one restricts Islamic patterns to patterns that are made in the period from 900 A.D. to 1500 A.D. (see definition above) there is no indication to show that at that time it was known, be it in the Muslim culture or elsewhere, that it was only possible to make 17 distinct patterns. Yet all, or to be a little modest at this stage, at least almost all of the 17 patterns can be found in Islamic patterns made in the period mentioned above. The question of whether absolutely all can be found has evoked considerable interest, and I will return to this later.

It is impossible to know exactly what has guided the design of Islamic patterns but one may assume that there has not been a conscious intention of producing all the 17 wallpaper patterns. However, one may also assume that in the making of the patterns some intuitive perception of to what extent they were similar or different has been present but there is no evidence to show that there has been an intention to make as many different patterns as possible. The fact that the variations on the Khatem Sulemani (Figure 1) is so much used in Islamic patterns will in a natural way produce the possible rotational symmetries. The star in Figure 3 has an eight-fold symmetry, and therefore its presence in a pattern will make it possible to obtain a pattern with rotational order four. The star in Figure 6 has a twelve-fold symmetry, so similarly this star could be part of a pattern with rotational order three or six. Of course both stars could be part of a pattern with rotational order two.
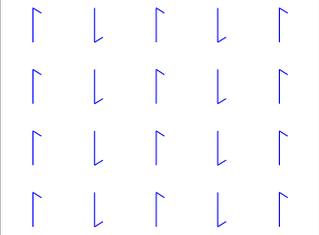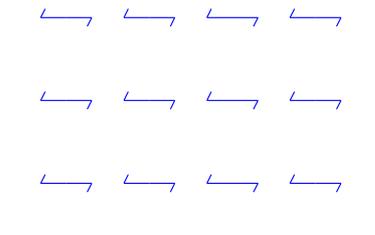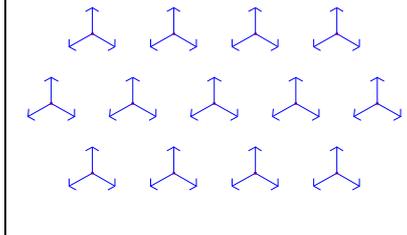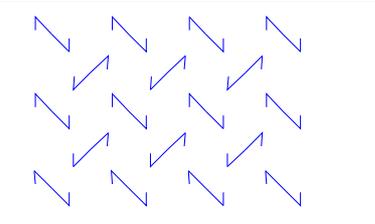

4. Patterns of the Alhambra

The palace Alhambra, outside of Granada in southern Spain, was built over a period of more than 100 years starting from the middle of the 13th century. This was a time when the south of Spain was dominated by the Moorish culture. Gradually the Moors were driven out by the Christians, and Alhambra was the last stronghold of the Moors. Eventually they had to abandon the castle in 1492. Many of the constructions from the time of the Moors in Spain have been destroyed but Alhambra is preserved. The same can to a large extent be said for the Great Mosque in Córdoba. This was also not destroyed but used for Christian purposes immediately after the Moors left, and much later a cathedral in baroque style was constructed in the middle of it. Despite the fact that the Great Mosque was turned into a Christian site, there are still many Islamic patterns that can be admired inside it, and of course the Alhambra is a very rich source of such patterns. Over the last 60-70 years many people have worked on getting an overview of the patterns of the Alhambra. These patterns most certainly fulfil criterion 1 for an Islamic pattern quoted earlier, namely that the pattern should be transcribed

with Arabic Calligraphy from the Quran. Recently there has been an extensive work going on connected to collecting and translating the writings in Arabic that can be found on the walls and ceilings of the palace, and according to an article in the Norwegian newspaper Klassekampen (Nash, 2009) the results of this work will soon be available on DVDs. I will not go into this topic here but concentrate on the mathematical content of the patterns, and in particular the classification of the patterns in terms of wallpaper groups.

The first serious work on this seems to have been done by Edith Müller who in her PhD thesis from 1944 identified 11 of the 17 possible groups in the decorations of the Alhambra (Grünbaum, Grünbaum, & Shephard, 1986). However, it has been claimed on several occasions that all 17 really can be found there. For example George E. Martin writes in his textbook *Transformation Geometry* that "all of the seventeen groups were implicitly known to the Moors in the decoration of the Alhambra" (Martin, 1982, p. 111). A similar assertion was made by Hermann Weyl in his classical text *Symmetry* when he writes: "Examples for all 17 groups of symmetry are found among the decorative patterns of antiquity, in particular among the Egyptian ornaments" (Weyl, 1952/1989, p. 103). As these assertions were set out without properly documenting that all 17 groups were actually represented, Branko Grünbaum started to work on the problem and made a thorough investigation of the patterns of the Alhambra. This resulted in a paper (Grünbaum, Grünbaum, & Shephard, 1986) where the authors claim to have identified 13 out of the 17. They also claim that the remaining four groups cannot be found in the Alhambra, and furthermore, that they "do not appear to be represented in other Moorish artefacts either" (p. 642). The four groups not found are *p1g* (often just denoted *pg*), *p2, p2gg* (often just denoted *pgg*) and *p3m1* (see Table 1). In another article Grünbaum (1984) also discusses ornaments in Egyptian art, and he claims that also there some of the symmetry groups are missing. Grünbaum's claim is that the group *p3m1* is represented neither in Egyptian nor in Moorish ornaments. Hilton and Pedersen (1984) in a commentary to Grünbaum's (1984) article mention a book on Japanese pattern design where also this particular group is missing.

Table 1. The four missing groups according to Grünbaum.

No rotation	Only 180° rotation	Only 120° rotation
p1g	*p2*	*p3m1*
	p2gg	

The assertion that a certain pattern does not exist within a certain culture is rather strong. Even when it comes to a reasonably limited space like the Alhambra this is a strong statement. However, even though the properties of the various wallpaper groups are clearly defined, it is not so obvious to determine to which group a given decoration from the Alhambra belongs. There are several issues to consider. One issue is that many of the patterns are filled with scriptures. Should they be ignored? The answer to this question will have to be 'yes', otherwise there will be very little symmetry left. Another issue is that many patterns in fact cover rather small areas, and to be counted as a genuine wallpaper pattern, they would have to be imagined to extend indefinitely in two directions. This may not always seem natural, as I will illustrate with an example later. A third issue is whether a pattern should be considered as Islamic (or Moorish) just because it can be found within the boundaries of the Alhambra. I will return to this question but first I will discuss two problems which are of greater interest from a mathematical point of view. I will illustrate these problems with two examples of patterns from the Alhambra, shown in Figures 8 and 9. See also (Grünbaum, 2006) for a discussion of these problems, and a critique against some of the claims that have been made that all 17 wallpaper groups really are present in the Alhambra.
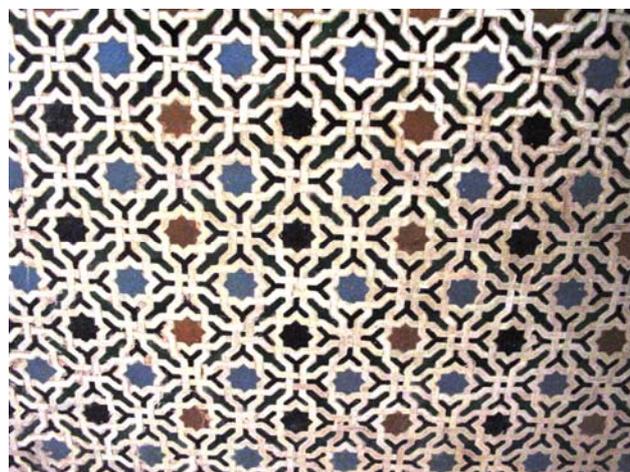

Figure 8

Figure 9

In Figure 8 one easily recognises the 12 pointed version of the Khatem Sulemani (Figure 7) which could give a hope of having a rotation of order six in the pattern. And indeed, were it not for the colours, the centre of each of the small black 12 pointed stars would have been a rotation centre of order six in the pattern. In addition, again ignoring colours, there are reflection axes through the centres of the stars in many directions. Hence, if one considers the black, blue and yellow parts as just being 'coloured', the pattern would be a *p6mm*. However, taking into consideration the different colours, the symmetries are by far not that abundant. The blue, pointed shapes that run horizontally in the pattern have the effect that the centre of the 12 pointed star is only a rotation centre of order two. There is also a centre of order two in the middle of each pair of the blue shapes, and there are horizontal and vertical reflection axes through the centres of the black stars. *With* colours the pattern is a *p2mm*. In Figure 9 one recognises the eight pointed Khatem Sulemani (Figure 4) which potentially could endow the pattern with an order four rotation. Without colours the centre of the eight pointed star would be a centre of order four, and the same would be the case for the centre of the cross shaped pattern between each Khatem Sulemani. The pattern would, ignoring colours, be a

p4mm. Taking the colours into consideration both these points reduce to centres of order two, leaving us also for this pattern just with a *p2mm.* However, there is another snag with this pattern, in addition to the colours. In Figure 10 I show a close up of a part of the pattern. Studying this carefully one can see that the white parts of the pattern are interlaced in an over/under manner. If this effect is considered, the reflection symmetries that otherwise would be there, disappear. The rotation centres of order two are preserved, however. This interlacing is a very common effect used in Islamic patterns, as is also documented in Figure 11 which is a picture from the Great Mosque in Córdoba. The pattern in Figure 9, potentially a quite rich *p4mm* will, taking both colours and interlacing into account, reduces to a mere *p2.*
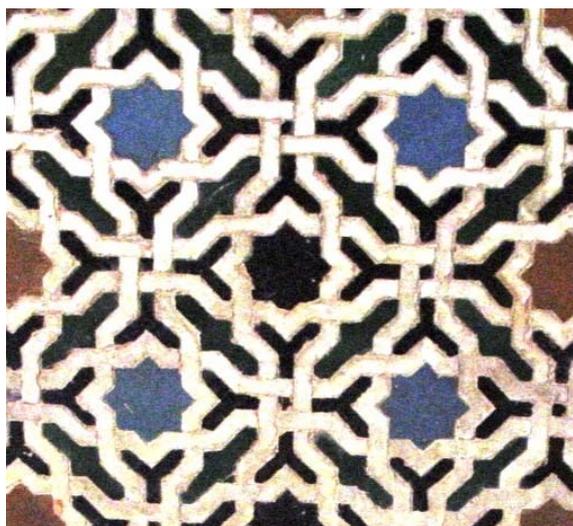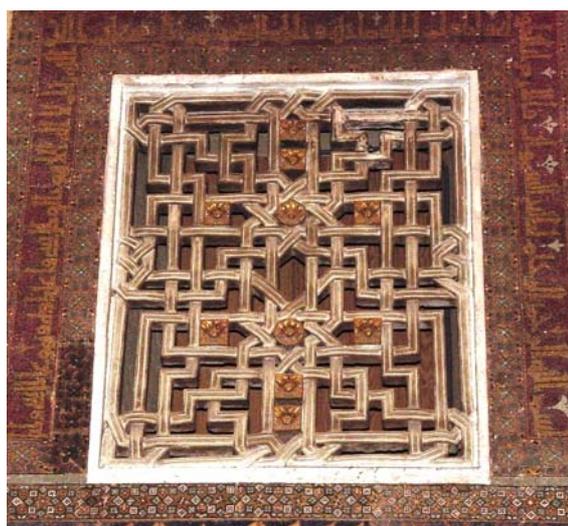

Figure 10

Figure 11

These examples show that before venturing into classifying patterns in symmetry groups certain decisions have to be made. Indeed, if one decides to count both colours and interlacing Figure 9 gives an example of one of the groups (*p2*) that Grünbaum et al. (1986) claim not to exist in the Alhambra. To show that it need not be so complicated I will show two examples of patterns that are found several places in the Alhambra and also elsewhere in Islamic art. Here I present them in a very clean version where there is no doubt as to their classification. The pattern in Figure 12 is clearly a *c1m* and the one in Figure 13 is a *p4.*

Figure 12

Figure 13

5. More about the four missing groups

After Grünbaum made his rather strong statements about the missing groups (Grünbaum, 1984; Grünbaum et al., 1986) the interest rose in determining whether it really was the case that there were only 13 different groups to be found or whether indeed all were present. In particular among Spanish mathematicians there was considerable interest in this question, and Pérez-Gómez (1987) published an article where he comes to the opposite result as Grünbaum. Pérez-Gómez claims, and accompanies his claim with examples, that the four missing groups in fact can be found in the Alhambra. The example of the group *p2* used by Pérez-Gómez has similar properties as my example in Figure 10, that reflections are lost because of interlacing. In the discussion about which patterns can be found there, a distinction has sometimes been made between what is in the Alhambra proper and what is in the museum of the Alhambra. For example, this has this been used to explain why Edith Müller did not find more than 11 groups because the museum may not have been accessible when she made her investigations (Grünbaum et al., 1986). The example by Pérez-Gómez of the *p2* is according to his article (Pérez-Gómez, 1987) to be found in the museum, whereas my example (Figure 10) being a *p2* under the same rules of classification is in the Alhambra proper. Grünbaum et al. (1986) seem to disregard interlacing which they refer to as *layered symmetry* defined as "an ordinary symmetry possibly combined with systematic interchanges of top and bottom layers at crossings" (p. 647). It is however possible to see the interlacing without going into three dimensions. See Figure 14 which is a plane representation of the central motive in Figure 10 with no reflection symmetries. Note that the centre of the star would be a centre of order four *with* the interlacing and *without* colours.
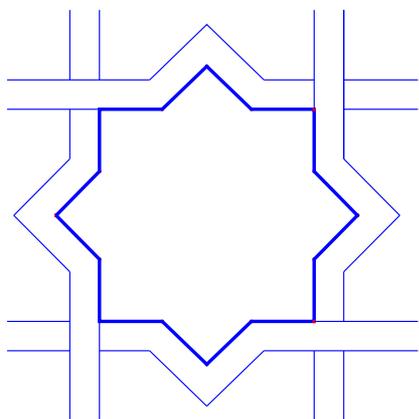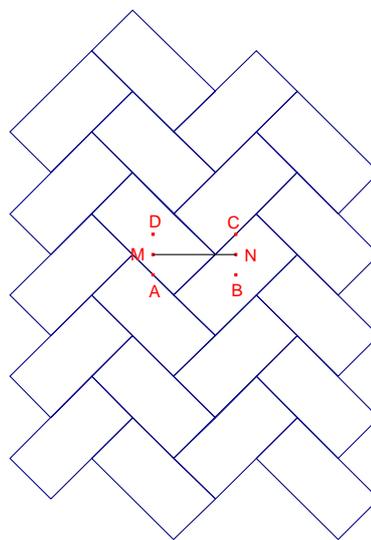

Figure 14

Figure 15

The missing *p2gg* is pointed to by Pérez-Gómez (1987) as represented in several floors of the Alhambra, and in the article it is depicted like in Figure 15. This pattern has a glide reflection given by the axis MN and the vector AB. Furthermore, it has a rotation centre of order two in the point C (or A). The question that could be raised here is whether this counts as an Islamic pattern. It is obviously a very common floor tiling, and presumably nobody would, just by looking at it, think of it as having any special connection to Islamic culture. The claim made by Grünbaum et al. is that "13 different wallpaper groups were identified among the symmetry groups of the ornaments found there" (1986, pp. 641-642). It can reasonably be

argued that a floor tiling is not an ornament. Pérez-Gómez' example of *p1g* is the same basic pattern as in Figure 15, also from a floor tiling, but with colours added, resulting in loss of symmetries.

I mentioned earlier that it sometimes could be somewhat artificial to see a pattern as extending indefinitely in two directions. As an example, look at the picture in Figure 16. This is a decoration from the Great Mosque in Córdoba. A pattern can here be recognised in the top segment and in four vertical strips. Each of these parts extended would certainly be a *p2mg* but to see this ornament as a *p2mg* one will have to focus on one of these parts and disregard the rest of the ornament. The same technique seems to be necessary to use also in connection with Pérez-Goméz' example of a *p3m1*.
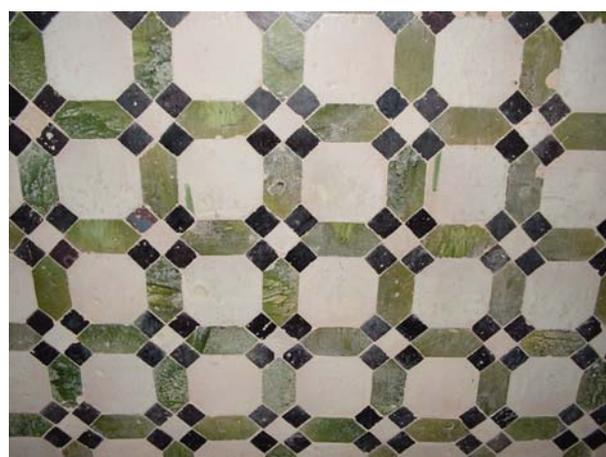

Figure 16

Figure 17

As a final example I want to point to the pattern in Figure 17, also from the Alhambra. As it is this pattern has several centres of order four and it has reflection axes in many directions. It is a *p4mm*. John Jaworski has written a very nice leaflet called *A Mathematician's Guide to the Alhambra* (Jaworski, 2006). In this leaflet he describes in detail how the 17 wallpaper groups are constructed, and he also gives arguments to show that there cannot be more than 17. Moreover, he discusses in particular the two patterns shown in Figures 8 and 16. He refers to these two as "The Mothers of All Patterns", and he claims that by colouring these two "genesis tilings" in a suitable way all 17 wallpaper groups can be found using only these two. The tiling in Figure 8 is especially rich but it can never produce fourfold symmetry. This is however accomplished by the pattern in Figure 17 which, on the other hand, never will give three- or sixfold symmetry. Naturally there will be some overlap in this construction in the sense that groups with rotation centres of order two and groups without rotational symmetry can be produced with both Figure 8 and Figure 17 as a "mother pattern".

6. Conclusion

It may not be a matter of great importance to give an ultimate answer to the question whether all 17 wallpaper groups can be found in the Alhambra or not. The discussion shows, however, that creating links between mathematical concepts or ideas and reference contexts is not as easy as one might immediately think. The mathematical concept for each of the symmetry groups is precisely defined, and its content is fully captured by the symbolic signs that I have used. For example the symbol *p4mm* contains, by convention, exactly the

properties of this particular group. By drawing, one could make an iconic sign (Peirce, 1998) for the same group but no matter how carefully this drawing is made it would only be an object with a certain likeness to the group in question. Taking instead of a drawing, a decoration like for example the one showed in Figure 9 the situation becomes much more complicated. Obviously, this decoration has certain likenesses to the concept that I am thinking about (the concept that I represent by *p4mm*) but it has also certain likenesses to other concepts, concepts that could be represented by *p2* or *p2mm*. Peirce writes that an icon relates to the object it stands for in a way that it "excites an idea naturally allied to the idea that object would excite" (1998, p. 13). It could be argued that the pattern in Figure 9 would excite ideas about *p4mm,* or *p2mm,* if we pay attention to the colours but not the interlacing. However, a strict analysis would give that this pattern really is a *p2*, and at the same time it might be reasonably argued that it does not give the impression of a *p2*. Looking at it from a distance one might not even notice the interlacing. To see this, compare the impression that you get by looking at Figure 9 and then at Figure 10.

I will conclude by asserting that the pattern in Figure 9 could be said to be a sign or a reference context for several wallpaper groups because it evokes ideas connected to both *p2, p2mm* and *p4mm*. So maybe an undisputable answer to the question about what symmetry groups are present in the Alhambra is impossible to give. I hope that through this article I have been able to give an impression of a strong link between mathematics and art, and to show that artistic expressions indeed can be a rich source for discussing interesting mathematical concepts.

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