Mathematical competence - what is it and what ought it be?

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1 Introduction

In this essay the author will take a closer look at the terms achievement and competence in the context of school mathematics. Previous research indicates that in some cases even high achieving students make use of superficial reasoning, lack a deep conceptual understanding and struggle with non-routine problems (Cox, 1994, Selden et al., 1994, Lithner, 2000...). Here the author uses the terms reasoning and understanding loosely, knowing well that they are aspects of internal, cognitive processes that is impossible to observe directly. Nevertheless, this creates an apparent paradox as the terms competence and achievement are obviously interrelated and it raises the question: aren't high achieving students also mathematically competent students? On the surface these might seem like a trivial question. Surely those students who do well in school are also competent students. High achievements mean that the students are mathematically competent. However, this inference rests on the premise that the socially defined label of high achievements corresponds with the concept of mathematical competence. Examining what the terms achievement and competence mean in a school mathematics context can shed some light on this question. To investigate more closely the underlying premise, the author intends to differentiate mathematical competence into a descriptive point of view and and a normative point of view using Hume's famous is-ought dichotomy.

Normative statements relate, in general, to an ideal standard or model. Often seen as how how things should be and it is inherently value laden. Things are good or bad, right or wrong. Descriptive statements are, on the other hand, an attempt at describing the world as it really is. They are factual statements about reality. Hume famously postulates in his *A Treatise of Human Nature* (1739) that:

> "In every system of morality, which I have hitherto met with, I have always remark'd, that the author proceeds for some time in the ordinary ways of reasoning, and establishes the being of a God, or makes observations concerning human affairs; when all of a sudden I am surpriz'd to find, that instead of the usual copulations of propositions, is, and is not, I meet with no proposition that is not connected with an ought, or an ought not. This change is imperceptible; but is however, of the last consequence. For as this ought, or ought not, expresses some new relation or affirmation, 'tis necessary that it shou'd be observ'd and explain'd; and at the same time that a reason should be given; for what seems altogether inconceivable, how this new relation can be a deduction from others, which are entirely different from it."

Hume highlights the problem of deriving how something *ought* to be from how something *is* vis-a-vis morality from a purely deductive perspective. Although mathematics education do include a number of issues related to morality and ethics, in this essay the focus is on what mathematical competence is and what it should be. Or, more specific, what is regarded as mathematical competence in the classrooms versus what is regarded as mathematical competence by the mathematics education research community. The focus of this essay is therefore the difference between descriptive and normative statements in more general, rather than being limited to issues of ethics and morality.

2 What competence is and ought to be

Normative statements are an integral part of humanity and found in all aspects of life; they are found in laws, traditions, ethics, parenting, sports etc. From *murder is wrong* to *song A is better than song B*. Similarly, we find numerous normative statements in the mathematics education community regarding mathematical competence. Ranging from issues regarding equity to more specific ideas on how and what students learn in school mathematics. In the following subsections a normative perspective on the concept of mathematical competence is given. However, the purpose is not to give a complete and correct definition of mathematical competence, but rather give the reader an idea of what different communites deem to be important elements of mathematical competence. The examples and definitions offered are not random, but could be replaced by others.

However, looking at it from a normative perspective needs some clarification. Defining what mathematical competence in a school setting should be, depends on context and purpose. Mathematical competence is a very broad term that encapsulates multiple dimensions. For instance, is mathematical competence looked from a purely theoretical perspective, in an attempt to extend mathematics as a research subject into a school setting or is mathematics looked upon more pragmatically as a tool the students need to participate in society. These perspectives in turn influence other dimensions of mathematical competence. Both in terms of the mathematical content the students should learn, but also the situations or context the content is located and the cognitive processes activated when the content is met. The epistemology of mathematics also influences the the definition of mathematical competence; is learning looked on from an aquisitionist or a participationist perspective. The basic tenet of aquisitionism is that the individual development proceeds from personal aquisitions of knowledge to participation in collective activities. Participationists, on the other hand, claim that taking part in collective activities precedes individual knowledge (Sfard, 2006).

2.1 Epistemology

Before a closer look at the specific details of mathematical competence is given, a short comment on the epistemology of mathematics itself is offered here. The purpose is simply to illustrate how learning theory can influence the definition of mathematical competence. Pisa, Programme for International Student Assessment, is an international project implemented by OECD to assess and compare 15 year olds proficiency and competence in mathematics, science, reading and problem solving skills in participating countries (OECD, 2009). The basis of PISA's definition of competence, is to what extent 15 year olds can be regarded as informed, reflective and intelligent citizens and consumers:

> "Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen."

Here, the framework uses the terms literacy and world to emphasise mathematical knowledge as a functional tool in which the individual makes use of and understands the natural, social and cultural world around us. The theoretical, or epistemological, foundation is mathematics as a language, used to mediate complex social activity. This implies that students must learn the "design features involved in mathematics discourse (the terms, facts, signs and symbols, procedures and skills to perform certain operations in specific mathematical sub-domains, and the structure of those ideas in each sub-domain)".

Harel (2008) offers a different perspective on mathematics. Mathematics is seen from the individual's perspective and is defines as an individual activity consisting of mental acts, way of thinking and way of understanding:

> "A person's statements and actions may signify cognitive products of a mental act carried out by the person. Such a product is the person's way of understanding associated with that mental act. Repeated observations of one's way of understanding may reveal that they share a common cognitive characteristic. Such a characteristic is referred to as a way of thinking associated with that mental act".

These two viewpoints are examples of the socio-cultural tradition (participationist) and the cognitive tradition (aquisitionist) (Sfard, 2006). From the individual constructivist perspective, aquisitionist, mathematics is primarily seen as individual knowledge. What do the individual understand and what mathematical knowledge does he or she possess. On the other hand, in socio-cultural "paradigm", or participationist, mathematics is primarily seen as a language, used to mediate complex social activity. The two viewpoints presented here are extremes on an axis and mentioned here only to serve as an illustration of how epistemology could influence the very definition of mathematical competence (William, 2007).

2.2 Math wars

The idea that mathematical competence somehow depends on certain premises such as epistemology, would be alien to many people. Mathematics are universally true. Unlike science, which deals in evidence and can be falsified, mathematics produce proofs that are universally true, independent of culture. Given a square, the formula for finding the area works, regardless of whether you are in Norway or China, now or 200 years ago. You either know mathematics or you don't. Yet, the topic of mathematical competence is to this day debated. From the more general issues such as equity vs. excellence to more specific issues regarding for instance conceptual understanding vs. procedural skill. These and others are all issues that mathematicians, mathematics educators and even politicians still debate. In recent years, for example, this seen during what has been referred to as the math wars (Schoenfeld, 2004). The debate was triggered with the publishing of *Curriculum and Evaluation Standards for School Mathematics* by the *National Council of Teachers of Mathematics*, NCTM, in 1989. On one side, you had proponents of reform, who wanted a greater focus on mathematical reasoning, problem solving and conceptual knowledge. On the other side, the supporters of "traditional mathematics teaching" wanted the curriculum to be centered around developing procedural skills. According to Schoenfeld (ibid), the debate also reflected a deeper divide, whether mathematics is for the elite or for the masses. "Traditional mathematics teaching", intentional or unintentional, was a perpetuation of privilege. It helped the rich stay rich and the poor stay poor. The reform curriculum, the standards, proposed by the NCTM could be seen "as a threat to social order". The debate quickly became political in nature, following the classic divide between progressivism and conservatism.

The math wars also indicated that there seemingly is a difference in the way the mathematics education community and the mathematics community view mathematical competence. In the midst of the math wars, the U.S. Department of Education released a list of recommended math books to the nation's 15000 school districts. Many of the books were based on reform mathematics. Within a month of the release, 200 university professors, including seven Nobel laureates and winners of the Fields Medal, added their names to an open letter asking the department to withdraw their recommendations (Klein, 2002). Furthermore, Schoenfeld (2004) states that mathematicians may be more comfortable with the core values cherished by the proponents of traditional mathematics education and Ralston (2004) writes that the math wars pitted research mathematicians against university mathematics educators and mathematics teachers.

It would probably be reasonable to assume that both sides in the math wars would concede to the fact that both "conceptual understanding" and "procedural skill" is part of the overall concept of mathematical competence. However, the focus of the curriculum; the perceived importance of understanding and skill, relative to each other, also influence one's view of mathematical competence. If one aspect of competence is deemed to be more important than another aspect, then the person would be more competent if the he or she masters the first and not the second and vice versa.

2.3 A definition

For the purpose of simplicity the author will focus here on mathematical competence in terms of cognitive processes as defined by the mathematics education research community. But first, another dimension of competence will quickly be mentioned, namely content. Mathematical content is usually organized into categories such as geometry, algebra, arithmetic, calculus etc and reflect historically well-established branches of mathematical thinking. For instance, in the Norwegian mathematics syllabus, we find content areas such numbers, geometry, economy, probability, algebra etc (Ministry of Education & Research, 2006).

There are several frameworks that similarly describe mathematical competence in terms of cognitive processes (see for instance OECD, 2009, NCTM, 2000, Neubrand et al., 2001 and others). Cognitive processes, in this context, describe mental and physical behavior when doing mathematics. Here, cognitive processes refers to both the mental and physical processes, activities and behaviors. In other words, the focus is what can the individual do. To exemplify mathematical competence in terms of cognitive processes, the framework presented in Niss and Jensen (2002) is used in this essay. As mentioned, other frameworks could have been used instead, but this framework defines mathematical competence as eight clear and distinct competencies, which makes it useful as a reference point. Niss and Jensen (IBID) divide the eight competencies into two groups. The first group are to do with the ability to ask and answer questions in and with mathematics:

- Thinking mathematically. Such as: posing questions that are characteristic of mathematics and knowing the kind of answers that mathematics may offer. Understanding and handling the scope and limitations of a given concept. Extending the scope of a concept and distinguishing between different kinds of mathematical statements.
- Posing and solving mathematical problems. Such as: identifying, posing and specifying different kinds of mathematical problems. Solving different kinds of mathematical problems.
- Modelling mathematically. Such as: analysing foundations and properties of existing models. Decoding existing models and performing active modelling in a given context.
- Reasoning mathematically. Such as: Following and assessing chains of arguments. Knowing what a mathematical proof is. Uncovering the basic ideas in a given line of argument. Devising formal and informal mathematical arguments.

The second group are to with the ability to deal with and manage mathematical language and tools:

- Representing mathematical entities. Such as: understanding and utilising different sorts of representations of mathematical objects. Understanding and utilising the relations between different representations of the same entity. Choosing and switching between representations.
- Handling mathematical symbols and formalisms. Such as: decoding and interpreting symbolic and formal mathematical language. Understanding the nature and rules of formal mathematical systems. Translating from natural language to formal/symbolic language. Handling and manipulating statements and expressions and containing symbols and formula.
- Communicating in, with and about mathematics. Such as: understanding others' written, visual or oral "texts" about matters having a mathematical content. Expressing oneself at different levels of theoretical and technical levels about such matters.
- Making use of aids and tools. Such as knowing the existence and properties of various tools and aids for mathematical activity and their range and limitations. Being able to reflectively use such aids and tools.

The obvious challenge that presents itself here, is recognizing, identifying and classifying the level of each competency for a specific individual. It would beyond the scope of this essay to evaluate and discuss every single competency in detail. So, and again for simplicity a closer look at one of the competencies, namely reasoning, will be taken in this essay. There are several frameworks and taxonomies that describe categorise mathematical reasoning. Here, a framework that the author has used himself in his own studies will be given as an example of how mathematical reasoning can be qualitatively classified. Lithner (2008) separates mathematical reasoning into two main categories; *creative reasoning* and *imitative reasoning.*

The basic idea of creative reasoning, or *creative mathematically founded reasoning* as it is also referred to in the framework, is the creation of new and reasonably well-founded task solutions. Not necessarily geniality or superior thinking. For the reasoning to be called creative reasoning, two conditions must be met (Bergqvist, 2007):

- The reasoning sequence must be new to the reasoner (novelty)

- The reasoning sequence must contain strategy choices and/or implementations supported by arguments that motivates why the conclusions are true or plausible (plausibility), and are anchored in intrinsic mathematical properties of the components involved in the reasoning (mathematical foundation).

Imitative reasoning is a term that describe several different types of reasoning which are based on previous experiences, but without any attempts at originality. This means that students try to solve problems and exercises by copying textbook examples, earlier task solutions or through remembering certain algorithms. Imitative reasoning is in many cases a superficial sequence of reasoning, not grounded on intrinsic mathematical properties, but rather on surface properties. The students chose their strategy for solving the problems on superficial properties they recognize from earlier experiences and not on intrinsic mathematical properties.

Imitation is, however, not intrinsically "wrong" or "bad". Vygotsky (1978) relates imitation to the zone of proximal development and general learning theory. If a solution to a problem or activity is within a child's zone of proximal development, the child can adopt it through imitation. Imitation becomes a key factor in learning of higher cognitive processes. Imitative reasoning can also reduce the cognitive load when solving mathematical problems. A professional mathematician may for instance make use of familiar algorithms when solving routine problems. Appropriate use of imitation or imitative reasoning is not "bad" or "wrong". Inappropriate use is, however "wrong" or "bad". In the context of solving mathematical problems, inappropriate is classified as when strategy choices are based on superficial mathematical properties, such as surface appearance, and not intrinsic mathematical properties. In other words, what the mathematics community would classify as incorrect and correct solutions.

2.4 What competence is

To give a complete description of the world as it is vis-a-vis mathematical competence in actual classrooms, is a daunting task. Therefore, a few findings from previous studies and the author's own work will be used to give an idea of the current state of affairs. To give some insight into the situation as it currently is. In a yet to be published study (Haavold, 2010), the relationship between the socially constructed "high achievements" in school mathematics and the theoretical concept of "mathematical competence" was investigated. The purpose of the study was to was to characterise high achieving students mathematical reasoning when given an unfamiliar trigonometric equation. As seen, mathematical competence is extremely complex and multi layered. Therefore, the investigation had to be simplified. The aim was to capture some key aspects of high achieving students' reasoning structure when working with a mathematical problem, which in turn could say something about high achieving students' general mathematical understanding. In the study, high achieving students were defined as students who consistently got grades five and six in upper secondary school mathematics. The empirical data in the study was collected from three clinical task based interviews. In each interview, the students were given a specific trigonometric task designed by the author and asked to solve it while they were "thinking aloud".

When the students were first given the problem "given $\sin x + \cos x = a$, find a", where a was given as a parameter, all three attempted what Lithner (2008) refers to as algorithmic reasoning. The students in varying ways attempted to find an algorithm or formula that would solve the equation. Algorithmic reasoning, when applied correctly, can reduce the cognitive load of solving mathematical problems. However, in this case, all three students attempted to use or find algorithms and/or formulas that were not helpful for solving the equation. A plausible explanation, is that the students did not consider the intrinsic properties of the equation, but focused instead on the surface appearance. On the surface, the equation looked like equations they had met earlier in the textbook. The students' behavior when they first approached the equation, reveal that imitation is a strong characteristic of their mathematical reasoning. All three students' first strategy choice, was to somehow simplify the equation using some standardized procedure or formula that they were previously familiar with. Only when they received some sort of guidance and help, were they able to solve the equation and display signs of mathematical creativity and high level reasoning. Based on the observations made in this study, it is the author's claim that the students possess the necessary domain knowledge to solve the equation. The students' were, for all intents and purposes, able to make the necessary connections and calculations to solve the equation on their own. The problem was a more general and structural behavioral pattern. When the students first began working on the equation, they immediately began looking for a formula, algorithm or procedure that would let them solve the equation. Later in the interview, all three students were able to focus on the intrinsic properties of the equation and solve it, but only with help. Although the results of this study are not generalizable to all high achieving students, the findings reinforce earlier findings, which have indicated that even high achieving students display superficial reasoning when faced with a mathematical problem (Selden et al., 1994, Lithner, 2000 & Schoenfeld, 1985).

Traditional mathematics teaching emphasizes procedures, computation and algorithms. There is little attention to developing conceptual ideas, mathematical reasoning and problem solving activities. The result is that students' mathematical knowledge is without much depth and conceptual understanding (Hiebert, 2003). These findings are seen in Selden et al.'s (1994) study where students with grades A and B struggle with non routine problems. Selden et al. concluded that the students possessed a sufficient knowledge base of calculus skill and that the students' problem solving difficulties was often not caused by a lack of basic resources. Instead, they say, traditional teaching does not prepare students for the use of calculus creatively. Lithner (2003) and Schoenfeld (1985) show how many of the students, even high achieving students, try to solve mathematical problems using superficial reasoning. A possible hypothesis which could explain this phenomenon is seen in Cox (1994), where the author argues that first year students in universities are able to get good grades by focusing on certain topics at a superficial level, rather than develop a deep understanding. Trends in International Mathematics and Science Study, TIMSS, Advanced 2008 (Mullis et al., 2009) and the PISA + study (Klette et al., 2008) show that this is an extensive problem in Norway. There is a lack of focus on problem solving and mathematical reasoning in both upper and lower secondary school. Students are rarely asked to explain their answers and communicate mathematical arguments to others. Instead, the primary activities in the classroom are direct instruction from the teacher and students working on problems on their own. Furthermore, the problems the students are working with are, according to Klette et al. (ibid), not stimulating problem solving skills.

3 Discussion

In the previous section the author gave a short overview of what competence is in classrooms and presented a few findings that show that in several cases there is a difference between high achievements and mathematical competence. That there is a difference between mathematical competence as defined by the mathematics education community and what is recognized as mathematical competence in the actual classrooms. This was in particular presented in the case of high achieving students' mathematical reasoning. The students got good grades and were recognized as "gifted" students by their teacher, yet when they were given an unfamiliar problem, their mathematical reasoning showed signs of undesirable characteristics. This leads to an interesting question. Why is there an apparent gap between the concept of mathematical competence as defined by the mathematics education community and what is considered to be high achievements in the classrooms?

3.1 Situated competence

The basic premise for the following discussion, is that students can only learn what they are given an opportunity to learn (Hiebert, 2003). Assuming the premise is true, the students' learning mileu must somehow limit the students' learning vis-a-vis mathematical competence as defined by the mathematics education community. To investigate this further, two theoretical models will be applied here. First, a closer look at how the curriculum is implemented in school will be given. Then, competence as participation in the classroom will be discussed.

Robitaille et al. (1993) gives a three tiered model of a curriculum. The first level is the intended curriculum, which represents the intentions, aims and goals of the curriculum. It is seen in national policies and official documents that represents the underlying philosophy and educational objectives. The second level is the content actually delivered by the teachers and is referred to as the implemented curriculum. What teachers teach, how they teach, which textbooks are used, how the textbooks are used, what activities are seen in the classroom etc. The third and last level is the attained curriculum, which is what the students have learned. One possible explanation for the apparent gap between the theoretical concept of mathematical competence and the socially constructed achievements in the classroom, could simply be that the intended curriculum do not reflect the mathematics education community's definition of mathematical competence. However, the competencies defined by Niss & Jensen (1999) are, to a large degree, seen in the intended curriculum, kunnskapsløftet, in school in Norway[1] (Ministry of Education & Research, 2006). Here competencies are referred to as basic skills. Some of the basic skills mentioned are: ability to form logical arguments, explain a way of thinking, posing questions, present a reasoned argument, formulate mathematical proofs, write/draw mathematical symbols and figures, use digital tools etc. The basic skills aren't an exact replica of the competencies, but there is a significant overlap and similarity between the two concepts. It is not unreasonable to conclude that the concepts are similar in terms of what constitutes mathematical competence. If the intended curriculum reflects the general idea of mathematical competence, then the apparent gap between high achievements and competence is caused by other factors.

Gresalfi et al. (2008) in their investigation of systems of competence in classrooms, problemize the assumption that competencies are attributes of individuals that can be externally defined. Instead, they propose a concept of individual competence as an attribute of a person's participation in an activity system. Here, what counts as competent depends entirely on context and situation. Opportunities for students to be understood as being competent depend on the tasks that they are assigned to work on, and on the agency and accountability with which they are positioned to do that work. This partly explains the gap between the theoretical concept of competence and the socially constructed high achievements in the classrooms. Students are considered mathematically competent because in that specific context/situation they have fulfilled their part of the didactical contract (Brousseau, 1997). The students have fulfilled the requirements set forward by teachers. This means that the learning mileu surrounding the students, with the teacher as the primary guiding force, determines what constitutes competence in the classroom. The result is that in many cases, mathematical competence in the classroom differs from mathematical competence in the curriculum.

3.2 The competence dilemma

In A treatise... Hume (1739) points out that how something ought to be can not be deduced from how something is. There is no valid inference from descriptive to normative statements. Although used by Hume in the context of morality and meta ethics, this observation is highly relevant within the field of mathematics education and to the concept of mathematical competence. In the previous sections, the author attempted to illustrate that

[1]The author of this essay is from Norway and therefore the focus is particularly on the Norwegian education system here.

there are several definitions and aspects of mathematical competence. Teachers and the mathematics education research community define mathematical competence differently. The research mathematics community and the mathematics education community define mathematical competence differently. Even within the mathematics education research community do we see different definitions of mathematical competence. The many dimensions of mathematical competence makes the concept highly complex and difficult to define. This raises several questions; arguably the most interesting one being, who is right?

However, if we do accept Hume's proposition as correct, none of them are correct, as you can not validly deduce how something ought to be from how something is. One of the arguments against Hume's proposition, is that ought can be derived through goal-directed behavior in the form of "if person A wants to achieve B, he/she ought to do C". For example, in the Norwegian subject curriculum it is stated that *"The mathematics in school contributes to developing the mathematical competence needed by society and each individual. To attain this, pupils must be allowed to work both theoretically and practically*" (Ministry of Education & Research, 2006). Similar statements are found in other frameworks and definitions of mathematical competence (see (Niss and Jensen, 2002), NCTM, 2000 etc). The goal is to develop competence needed by both the society and the individual, and to do so, certain "behavior" is required. There might be an empirical basis for this link, that in order to achieve B, he or she ought to do C. However, this creates another problem, as the goal B in itself is now defined by a normative statement. Ernest (1991) identifies five broad purposes for mathematics education:

- Acquiring basic mathematical skills, numeracy and social training in obedience.
- Learning basic skills and learning to solve practical problems with mathematics and information technology.
- Achieving understanding and capability in advanced mathematics, with some appreciation of mathematics.
- Gaining confidence, creativity and self-expression through mathematics.
- Empowerment of learners as critical and mathematically literate citizens in society.

The five broad purposes are also, in other words, the five broad possible goals for mathematics education. However, as follows by using Hume's proposition, and as Niss (1993) also points out, each goal or purpose is a normative statement. There is no way, outside essentially value arguments, to justify the purposes or goals for mathematics education. None of the purposes are more "correct" than the others.

In the introduction, the author set forth to investigate the terms high achievements and mathematical competence. By looking at it using Hume's dichotomy of normative and descriptive statements, and how it is impossible to infer how something ought to be from how something is, the mismatch between high achievements and mathematical competence is not unexpected. Mathematical competence is normative by nature and depends on the context. Context is here seen very broadly, encompassing many dimensions from the purpose of mathematics education, seen earlier, to more specific factors that influence one's behavior.[2]

References

Bergqvist, E. (2007). Types of reasoning required in university exams in mathematics. *Journal of Mathematical Behavior*, 26:348–370.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Kluwer Academic Publishers.

Cox, W. (1994). Strategic learning in a-level mathematics? *Teaching Mathematics and its Applications*, 13:11–21.

[2]See for instance Ernest's (1989) discussion of espoused and enacted beliefs.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In *Mathematics teaching: The state of the art*. London: Falmer Press.

Ernest, P. (1991). *The philosophy of mathematics education*. London: Falmer Press.

Gresalfi, M., Martin, T., Hand, V., and Greeno, J. (2008). Constructing competence: an analysis of student participation in the activity systems of mathematics classrooms. *Educational studies in mathematics*, 70(1):49–70.

Haavold, P. (2010). What characterises high achieving students' mathematical reasoning. In *The elements of creativity and giftedness in mathematics*. Sense Publishers.

Harel, G. (2008). What is mathematics? a pedagogical answer to a philosophical question. In Gold, R. and Simons, R., editors, *Current Issues in the Philosophy of Mathematics From the Perspective of Mathematicians*.

Hiebert, J. (2003). What research says about the nctm standards. In *A research companion to Principles and Standards for school mathematics*. National Council of Teachers of Mathematics, Reston, VA.

Hume, D. (1739). *A Treatise of Human Nature*. London: John Noon.

Klein, D. (2002). A brief history of american k-12 mathematics education in the 20th century. In *Mathematical cognition: A volume in current perspectives on cognition, learning and instruction*. Information age publishing.

Klette, K., Lie, S., Ødegaard, M., Øystein Anmarkrud, Arnesen, N., Bergem, O. K., and Roe, A. (2008). Pisa+: Lærings- og undervisningsstrategier i skolen. Technical report, ILS, UiO.

Lithner, J. (2000). Mathematical reasoning in task solving. *Educational Studies in Mathematics*, 41(2):165–190.

Lithner, J. (2003). Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics*, 52:29–55.

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, 67:255–276.

Ministry of Education & Research, K. (2006). *Kunnskapsløftet*. Oslo: Utdanningsdirektoratet/Norwegian directorate for education and training. Retrieved August 30, 2010 from http://www.udir.no/grep.

Mullis, I., Martin, M., Robitaille, D., and Foy, P. (2009). *TIMSS Advanced 2008 International Report: Findings from IEA's Study of Achievement in Advanced Mathematics and Physics in the Final Year of Secondary School*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

NCTM (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Neubrand, M., Biehler, R., Blum, W., Cohors-Fresenborg, E., Flade, L., Knoche, N., Lind, D., Loding, W., Moller, G., and Wynands, A. (2001). Grundlagen der ergänzung des internationalen oecd/pisamathematik-tests in der deutschen zusatzerhebung. *Zentralblatt for Didaktik der Mathematik*, 33(2).

Niss, M. (1993). Assessment in mathematics education and its effects. In *Investigations into assessment in mathematics education: An ICMI study*. Dordrecht, The Netherlands: Kluwer.

Niss, M. (1999). Aspects of the nature and state of research in mathematics education. *Educational studies in mathematics*, 40:1–24.

Niss, M. and Jensen, T. H. (2002). *Kompetencer og matematiklæring. Ideer og inspiration til udvikling af matematikundervisning i Danmark*. Utdannelelsesstyrelsens temahæfteserie nr. 18-2002. Undervisningsministeriet.

OECD (2009). Pisa 2009 assessment framewrok. key competencies in reading, mathematics and science. Technical report.

Ralston, A. (2004). Research mathematicians and mathematics education: A critique. *Notices of the American Mathematical Society*, 51(4):403–411.

Robitaille, D., Schmidt, W., Raizen, S., Britton, E., and Nicol, C. (1993). *Curriculum frameworks for mathematics and science*. Vancouver: Pacific Educational Press.

Schoenfeld, A. (1985). *Mathematical Problem Solving*. Orlando, FL: Academic Press.

Schoenfeld, A. (2004). The math wars. *Educational Policy*, 18:253–286.

Selden, J., Selden, A., and Mason, A. (1994). Even good calculus students can't solve non-routine problems. In Kaput, J. and Dubinsky, E., editors, *Research issues in undergraduate mathematics learning*. Washington, DC: Mathematical Association of America.

Sfard, A. (2006). Participationist discourse on mathematics learning. In *New mathematics education research and practice*. Rotterdam, The Netherlands: Sense Publishers.

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

William, D. (2007). Keeping learning on track. classroom assessment and the regulation of learning. In *Second handbook of research on mathematics teaching and learning*. Information age publishing.