ANALYZING BENARDETE'S COMMENT ON DECIMAL NOTATION

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Abstract. Philosopher Benardete challenged both the conventional wisdom and the received mathematical treatment of zero, dot, nine recurring. An initially puzzling passage in Benardete on the intelligibility of the continuum reveals challenging insights into number systems, the foundations of modern analysis, and mathematics education.

Keywords: real analysis; infinitesimals; decimal notation; procedures *vs* ontology

1. Introduction

Philosopher José Benardete in his book *Infinity: An essay in metaphysics* argues that some natural pre-mathematical intuitions cannot be properly expressed if one is limited to an overly restrictive number system:

> The intelligibility of the continuum has been found–many times over–to require that the domain of real numbers be enlarged to include infinitesimals. This enlarged domain may be styled the domain of continuum numbers. It will now be evident that $.9999\ldots$ does not equal 1 but falls infinitesimally short of it. I think that

> .9999... should indeed be admitted as a *number* ... though not as a *real* number. [Benardete 1964, p. 279] (emphasis in the original)

To a professional mathematician, Benardete's remarks may seem naive. The equality between 1 and 0.999.... is not in the realm of speculation, but rather an established fact. This follows from the very *definition* of 0.999... as the *limit* of the sequence 0.9, 0.99, 0.999, ... where the limit of a sequence (u_n) is defined, following any calculus textbook, as the real number L such that for every $\epsilon > 0$ there exists an $N > 0$ such that if $n > N$ then $|u_n - L| < \epsilon$ (and even in the hyperreal number system they are still equal, as Bryan Dawson recently explained in detail [Dawson 2016]).

It patently follows from the definition that 0.999... equals precisely 1 because the value $L = 1$ satisfies the definition stated, and there is nothing more to discuss. Or is there?

2. Vicious circle

As is patently evident from the definition given above, the procedure of taking the *limit* is real-valued by definition. If one is to treat Benardete's comment charitably, one cannot presuppose the answer to his query, namely that the number in question is necessarily a real number. The equality should, if possible, be conceived independently from a conception of 0.999... in terms of limits that presupposes that limits are, by definition, real-valued.

While the equality $0.999\ldots = 1$ (whether it results from a definition or from a mathematical demonstration) is absolutely necessary in order to ensure the coherence of the algebra with the real numbers, in some approaches to the real numbers the identification of $0.999\ldots$ with 1 is itself part of the *definition* of the real number line rather than a *theorem*. Thus, the real numbers can be defined in terms of unending strings usually referred to as their *decimal expansions*. In this approach to the real numbers, the strings $1.000\ldots$ and $0.999\ldots$ are *postulated* to be the same number, i.e., they *represent* the same real number; more generally, the equivalence relation on such formal strings is defined in such a way that each terminating string is equivalent to the related one with an infinite tail of $9s$.

In this approach to the real numbers, it is indeed correct to assert that the equality (in reality masking an equivalence) is not a theorem but rather a definition. It can be debated whether practically speaking this approach is a good approach to the real numbers; arguably it is for some purposes, but not for others.

Historically the first scholar to recognize that it is useful to represent numbers by *unending* decimals was Simon Stevin already in the 16th century, before the golden age of the calculus, and even before the symbolic notation of Vieta; see [Katz & Katz 2012] as well as [Błaszczyk, Katz & Sherry 2013, Section 2]. The identification of the two strings is first found in the second half of the 18th century in the work of Lambert (1758) and Euler (1765).

3. Other meanings

Can one assign any *meaning* to the symbolic string "$0.999\ldots$" other that defining it to be 1? That question cannot be answered without analyzing what *informal* meaning is assigned to $0.999\ldots$, *prior to* interpreting it in a formal mathematical sense.

Beginning calculus students often informally describe this as "nought, dot, nine recurring" or alternatively "zero, point, followed by infinitely many $9s$." The second description may not necessarily refer to any sophisticated number system like the real number system (e.g., equivalence classes of Cauchy sequences of rational numbers expressed in Zermelo–Fraenkel set theory), since at this level the students will typically not have been exposed to such mathematical abstractions, involving as they do equivalence classes of Cauchy sequences, Dedekind cuts, or other techniques from analysis.

It is also known that at this level, about 80 percent of the students feel that such an object necessarily falls a little bit short of 1. The question is whether such intuitions are necessarily erroneous, or whether they could lend themselves to a mathematically rigorous interpretation in the context of a suitable number system.

Rob Ely argues that such intuitions are not necessarily mathematically erroneous because they can find a rigorous implementation in the context of a hyperreal number system, where a number with an infinite tail of 9s can fall infinitesimally short of 1 [Ely 2010]. Namely, if H is an infinite hypernatural, then the hyperfinite sum $\sum_{n=1}^{H} \frac{9}{10^n} = 0.999\ldots9$

contains H occurrences of the digit 9, and falls short of 1 by the tiny amount $(0.1)^H$.

Infinite hypernaturals like H belong to a hyperreal line, say ${}^*\mathbb{R}$, which obeys the same rules, namely the rules of real-closed ordered fields, as the usual real line $\mathbb{R}$, and they behave in ${}^*\mathbb{R}$ as the usual natural numbers behave in $\mathbb{R}$, by the *transfer principle*.[1] This is in sharp contrast with Cantorian theories where neither cardinals nor ordinals obey the rules of an ordered field. Thus, there is no such thing as $\omega - 1$, while $\aleph_0 + 1 = \aleph_0$. On the other hand, according to Tirosh and Tsamir, students tend to attribute properties of finite sets to infinite ones:

> Research in mathematics education indicates that in the transition from given systems to wider ones learners tend to attribute the properties that hold for the former to the latter. In particular, it has been found that, in the context of Cantorian Set Theory, learners tend to attribute properties of finite sets to infinite ones–using methods which are acceptable for finite sets, to infinite ones. [Tirosh & Tsamir 2013]

The hyperrational number $\sum_{n=1}^{H} \frac{9}{10^n} = 0.999\ldots9$ can be visualized (virtually) on a hyperreal numerical line in the *halo* of 1 by means of

[1]The *transfer principle* is a type of theorem that, depending on the context, asserts that rules, laws or procedures valid for a certain number system, still apply (i.e., are "transfered") to an extended number system. Thus, the familiar extension $\mathbb{Q} \hookrightarrow \mathbb{R}$ preserves the properties of an ordered field. To give a negative example, the extension $\mathbb{R} \hookrightarrow \mathbb{R} \cup \{\pm\infty\}$ of the real numbers to the so-called *extended reals* does not preserve the properties of an ordered field. The hyperreal extension $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ preserves *all* first-order properties, such as the identity $\sin^2 x + \cos^2 x = 1$ (valid for all hyperreal x, including infinitesimal and infinite values of $x \in {}^*\mathbb{R}$). For a more detailed discussion, see the textbook *Elementary Calculus* [Keisler 1986].

Keisler's microscope, with the difference being 10^{-H}. To return to the real numbers, we have the equality $\mathrm{st}(0.999\ldots9) = 1$ where "st" is the operation of taking standard part (shadow).[2]

Thus, there are many hyperreal numbers representable as 0 followed by infinitely many 9s, e.g., $0,999\ldots9000\ldots$ (where the zeros start from some infinite rank), $0.999\ldots9123123\ldots$, $0,999\ldots999\ldots$, etc. Among such numbers only one is the natural extension of the usual sequence 0.(9) (unending 9s) and this indeed equals 1. The matter was explored in more detail by [Lightstone 1972].

The hyperrational number $\sum_{n=1}^{H}\frac{9}{10^n} = 0.999\ldots9$ is a *terminating* infinite string of 9s different from the one usually envisioned in real analysis, but it respects student intuitions. Intuitions like these underlie Leibniz's and Peirce's conceptions of the continuum. They can be helpful in learning the calculus, as argued in [Ely 2010]; see also [Vinsonhaler 2016], [Katz & Polev 2017].

The pedagogical issue is a separate one but what could be emphasized here is that the existence of such an interpretation suggests that we indeed do *assume* that such a string represents a *real* number when we prove that it necessarily equals 1. The idea that such an assumption can be challenged is in line with Benardete's comment.

[2] To elaborate further, note that each hyperreal r can be represented by means of its extended decimal expansion $r = \pm n.r_1r_2\ldots r_i\ldots r_H\ldots$ where $n \in {}^*\mathbb{N}\cup\{0\}$ and $r_i \in \{0,1,2,...,9\}$ for $i \in {}^*\mathbb{N}$.

4. Conclusion

It is well-known that the real number system provides an adequate foundation for mathematics. Yet the real number system does not explain the well-known phenomenon of student unease about the identity $1 = 0.999\ldots$ (nor the philosopher's unease).

It is similarly well-known that a set-theoretic justification of a hyperreal number system involves more work than is justified by the need of explaining $0.999\ldots$ to the students. Yet the crucial distinction between set-theoretic justification (ontology), on the one hand, and the procedures involving infinitesimals, on the other, has been emphasized in [Bascelli et al. 2016]. The pioneers of analysis like Leibniz, Euler, and Cauchy were already working with infinitesimals and exploiting the procedures based on them to great scientific effect. What we show in our paper is that such infinitesimal procedures can be used to address both the student unease and the philosopher's unease. Leibniz did not develop the set-theoretic ontology for infinitesimals simply because set theory did not exist yet, but he would have been more open to an infinitesimal gap between $0.999\ldots$ and 1 than the traditionally trained mathematicians today.

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