WITTGENSTEINIAN PEDAGOGY FOR MATHEMATICS

Rule-following, and why it matters for mathematics teaching and learning

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Abstract

Wittgenstein's work on rule-following and language games is well-known within philosophical circles. His work on the foundations of mathematics is also celebrated and contested with equal measure. This paper seeks to apply some of Wittgenstein's core philosophical writings on rule-following to justify a very particular type of mathematical pedagogy for mathematics and its teaching. It will be argued that since mathematics is fundamentally rule-like in nature, that Wittgenstein's work on rule-following directly applies. Other key elements of Wittgenstein's philosophy will also be central to this argument. For example, the notion of training, and of language-games, as well as what it means to engage in a meaningful way in these language-games. It will be argued that a rule in itself cannot guide, just like a signpost can point in a direction, but it cannot ensure that people will follow it correctly. So too in mathematics, it will be shown that the rules of mathematics can be interpreted in infinitely many ways, so that teaching mathematics as rule-like only would lead to inherent problems. As such, the case for some other guiding faculty will be made, and this, it will be shown, is the core justification for teacher-centred pedagogy in mathematics.

1 Introduction

Ludwig Wittgenstein was a failed elementary school teacher, in the main due to his lack of patience for those he was charged with educating. Despite, by all accounts, being regarded as a gifted teacher in terms of teaching content, he regularly had issues around his treatment of children, until his eventual resignation in April 1926 (Monk, 1991, pp. 224 & 232-3). At the age of seventeen, he was himself expelled from school for writing a controversial essay about the immortality of the soul. It should come as a surprise, therefore, that educators might have anything of any real meaning to learn from him. Moreover, in Wittgenstein's vast writings on philosophy, there is but one fleeting explicit reference to educators: 'Every explanation has its foundation in training. (Educators ought to remember this.)' (*Zettel*, §419).

2 Training

It is this concept of *training* which is the first interest in this paper. So significant was the concept of training to Wittgenstein, he warns that all explanations begin within it. What Wittgenstein meant, precisely, by training is a topic of widespread debate. He may have meant acquiring the ability to follow rules *in a similar way* to those who train the trainee, and simply obey orders as shown to us by an expert master. This notion may well have been entirely conceptual, rather than overtly literal (Glock 1996; Ryle 1949; Monk 2004). Alternatively, it may well be, despite his own contestations, that Wittgenstein was, in fact, a behaviourist in disguise. Wittgenstein himself was cautious to consider the idea:

> 'Aren't you nevertheless a behaviourist in disguise? Aren't you nevertheless basically saying that everything except human behaviour is a fiction?' – If I speak of a fiction, then it is of a *grammatical* fiction.
>
> (*Philosophical Investigations*, §307)

Adopting this view of Wittgenstein may lead us to conclude that his understanding of training was inexorably liked to simply changing behaviours, since, if he was indeed a behaviourist in disguise, behaviours would be either: (1) all that existed (metaphysical behaviourism) or, (2) all that mattered (methodological behaviourism) or, (3) logically identified with everything else (on the 'inside', for example) and so all that should be spoken about (logical behaviourism).

Some insight may come from the German word which Wittgenstein used when referring to what is translated to 'training' in the English translations of his works. The word in question is '*Abrichtung'*. Some German dictionaries use the word '*Dressur*' as a synonym for Wittgenstein's word. Literally translated, it is a *particular type training* to which Wittgenstein is referring. *Abrichtung*, for example, is often the word used in relation to training animals e.g. a dog, to carry out a particular action under guidance or instruction. It is safe to say that educators may not take too kindly to Wittgenstein suggesting that students should be trained like animals! The synonym word *Dressur* is translated more cautiously to refer to the process of getting something fit for a particular purpose. It entails a dimension of preparation *by training* for carrying out a specific role.

These words might suggest that Wittgenstein did not seek to deny the existence of an inner, rather that there are *grammatical* considerations we must take into account in how we speak about how people (in particular students) learn how to do anything of any real meaning. For Wittgenstein, there is a categorial difference between the first- and third-persons, and consequently, this category difference points to a difference in the *grammar* of how we speak in each case: in the first-person, we make non-criterial *avowals*, whereas in the third-person case we make criterial-dependent observations – and subsequently *report* on these observations using *descriptive* language – based on behaviour (Harré & Tissaw, 2005, p. 190). As Wittgenstein himself notes:

> Psychological verbs [are] characterized by the fact that the third person present is to be identified by observation, the first person not. Sentences in the third person of the present: information. In the first person present: expression.
>
> (*Remarks on the Philosophy of Psychology,* §63)

Wisdom (1967, p. 361) further captures this concept thus:

> The asymmetrical logic of statements about the mind is a feature of them without which they would not be statements about the mind, and that they have this feature is no more a subject suitable for regret than that fact that lines, if truly parallel, don't meet.

As Glock (1996, p. 56) notes, Wittgenstein rejects Carnap's argument that first-person psychological propositions can be confirmed or disconfirmed by a form of introspection or self-observation. For Wittgenstein, 'it does not make sense to verify a proposition like 'I am sad' by observing one's own posture and behaviour' (Glock, 1996; *Philosophical Reflections 89-90; Zettel* §539). To be clear, as Glock also observes, citing Wittgenstein, statements like 'I am in pain' are not in any way grammatically similar to statements like 'I am manifesting pain behaviour' (Glock, 1996, p. 56; *Philosophical Investigations* §244). This difference in grammar points to a difference in the nature of the statements in question.

Wittgenstein's position, therefore, rather than being behaviourism in disguise, is rather better understood to be fundamentally anti-Cartesian. In defining the grammatical differences between the first- and third-person language that we employ in how we talk about behaviour, it is not a denial of anything 'inner' or mental; rather it is a different way of representing the concepts which Wittgenstein took to be incorrectly represented in Cartesianism (Bax, 2011, pp. 40-1).

It seems, therefore, that Wittgenstein is not making any effort to deny that there exists something which lies *behind* the behaviour on show. He makes this clear in two other propositions from *Philosophical Investigations:*

> It's not a Something, but not a Nothing either! The conclusion was only that a Nothing would render the same service as a Something about which nothing could be said. We've only rejected the grammar which tends to force itself on us here.

> The paradox disappears only if we make a radical break with the idea that language always functions in one way, always serves the same purpose…
>
> (*Philosophical Investigations,* §304)

Furthermore, in §305, Wittgenstein goes further to claim 'What gives the impression that we want to deny anything?' opining that he is against only the idea that the picture of an inner process might allow us to use words correctly, and across contexts with no consideration as to the first- and third-person asymmetry which governs them.

This profound asymmetry, which is at the heart of Wittgenstein's philosophy of language, lets us see why *training* is so central to his philosophy, and why his ideas about how we learn to do any meaningful action – or to play any 'game' – is not a behaviourist notion, as opposed to being one founded on the philosophical role which language plays in how we communicate unambiguously about our observations (third-person) and avowals (first-person). Training lies at the foundation of this – to the extent that Wittgenstein claims it is the foundation of every explanation – as a process which *initiates* learners into the rules of the game being played. Wittgenstein is not suggesting that there are no moments of personal contemplation which might lead to a particular behaviour being made manifest; rather he is outlining that to acquire the ability to engage meaningfully in *any* type of behaviour, we must first be inculcated into the manner of acting which is in keeping with other similarly trained people.

Training, or '*Abrichtung'*, seems apt in this context; so that we might prepare students to engage meaningfully with the game, by first knowing how the game is played. As Malcolm (1986, p. 156) notes, 'If there is no *we* – if there is no agreement among those who have had the same training, as to what are the correct steps … then there would be no *wrong* steps, or indeed any *right* ones.' Training, therefore, is a blind acceptance of the rules, *initially,* so that we can engage meaningfully in the games we play. This is not a denial of one's rationality, nor is it an acceptance of *obedience* as mere *compliance*. Rather, it is an acceptance of the notion that in order to learn how to play a game, one must first grasp the rules, *and then* play the game *in agreement* with this rules. In this absence of this *agreed* set of rules to which we are exposed in our training, as Malcolm observes, correctness and incorrectness are no longer meaningful concepts; something we would surely want to avoid in *any* educational process. To put it another way, playing the game of 'learning' in isolation or atomistic form is not intelligible; so we acquire the rules of active participation in the game in our *training*, and thereafter engage meaningfully in accordance with these rules of play.

3 Language-Games

The concept of a language-game is also central to Wittgenstein's philosophy. Wittgenstein viewed language and games as analogically connected, in a similar way to how previous philosophers, like Frege and Russel viewed language as connected to a calculus or formal system of rules (Glock, 1996, p. 193). The game analogy which Wittgenstein often invoked was chess. The reason for this was to elicit the notion that language, like the game of chess, has rules. The rules of language are grammatical rules. The purpose of the rules of any game is to fix *what makes sense* within the game whilst playing it. The other analogical extension which Wittgenstein makes between language and games is that when playing a game, correct moves are established by *performing* the move. So too in language, *meaning is found in the use of the words* (*Philosophical Investigations,* §1, 9, 10, 20, 30, 35, 40, 41, 43, 120, 138, 139, 197, 247, 532).

It is crucial to understand that the notion of a language-game does not apply only to the use of words, for example. Rather, Wittgenstein is creating a model of understanding how human beings engage in *any* activity in a meaningful manner. Any behaviour which is to be understood in any meaningful sense must be constrained by a pre-existing set of rules which govern meaningful action. Playing a language-game requires an appreciation for the rules – attained in one's *training* – and then knowing how to act in agreement with those rules. The genius of Wittgenstein's invoking of the language-game analogy is found mainly in coming to appreciate that he does not insist that rules *in themselves* are sufficient to play any game. That is, a rule *in itself* cannot guide to its appropriate use. We can interpret a rule in an infinite number of ways. This gives rise to the notion of a *paradox* in how we might follow rules:

> This was our paradox: no course of action could be determined by a rule, because every course of action can be brought into accord with the rule. The answer was: if every course of action can be brought into accord with the rule, then it can also be brought into conflict with it. And so there would be neither accord nor conflict here.
>
> (*Philosophical Investigations*, §201)

As a consequence, a rule is like a *signpost* (*Philosophical Investigations*, §85); and part of learning how to play the game, is learning how to follow a signpost, to know which way it points, and to follow its direction. This following is an active process. It is something which means that in order to meaningfully engage in any activity (of educational value in particular) is to learn how to follow the rule, to be guided by the signpost *correctly.* As Wittgenstein opines:

> That's why 'following a rule' is a practice. And to think one is following a rule is not to follow a rule. And that's why it is not possible to follow a rule 'privately': otherwise thinking one was following a rule would be the same as following it.
>
> (Philosophical Investigations, §202)

This signpost-like guidance cannot, however, be a private possession. Indeed, one of the key elements of Wittgenstein's philosophy of language-games was the idea that we cannot develop a semantically meaningful language-game in isolation from other players. This idea is now celebrated as the *Private Language Argument* (*Philosophical Investigations,* §243, 246, 256, 269), and has been written about by many distinguished philosophers, most notably Kripke (1982) and Baker & Hacker (2014). The notion is a simple one, but has profound implications for the nature of how we engage in any activity with semantic considerations. Wittgenstein contests that language, as he defines it, cannot be conceptually private; that is, it cannot be something to which only one person has access.

Indeed, a fundamentally private language would, by definition, be inaccessible to anyone other than the person who 'owns' it. As such, this would mean that the person who owns the private language could not understand it themselves! Why? Well, if the person could understand their own private language, then by default – as a language user of other language-games which other people *can* play – the owner of the private language could translate the language into something which other language-users (players) could understand, hence violating its private nature. This is clearly absurd, and so it must follow that the private language cannot be understood by the person who claims to have private possession of it. This is clearly also absurd, since what would it mean to have possession of a private language which no-one, including the owner, can understand or use? Wittgenstein thus concludes that we do not play language-games in private, rather we engage in meaningful linguistic exchange with other users of the language, and all of the users are playing by the same set of agreed rules, shown to them in their *training*.

The implications of Private Language are far-reaching for our considerations of what it means to engage in learning anything meaningful within education. If we set learning languages, sciences, mathematics, arts, history, and religion, for example, inside this context, we begin to realise that *all learning* is a fundamentally community-based practice, which leans heavily on the notion of *training*. Wittgenstein, to be clear, is not saying that we cannot make our own contributions to these studies by personal reflection and moments of creativity. Rather, our entire engagement with any meaningful exchange of ideas is set inside a fiduciary framework of established rules and routines, which are public possessions, not private ones. Individual dissent from these rules, norms and maxims is discouraged *in the first instance*, in order to learn how to play the game *at all.* Becoming a *public* language user means developing, first, an appreciation of the rules of the language-game, and then *using* the rules in agreement with the other language-users who participate in the game.

4 Rule-following and Continuing Sequences

When we consider Wittgenstein's language-games analogy, and begin to note the centrality of rule-following in the development of these ideas, it becomes clear that for Wittgenstein, learning any new skill or ability can be considered as a process of learning the rules of the game (*training*) followed by developing an insight as to *when* and *how* to apply the rules in agreement with other language users. We have seen how Wittgenstein warns that simply knowing the rules is not sufficient for becoming a sophisticated language-user or game-player. Indeed, rules are the *signposts* which point the players of the game in the correct direction, but signposts only take us so far. So, for example, knowing that the knight moves 3+1 in chess does not ensure that we will move the piece correctly when in-game play. We must demonstrate *with regularity* whilst playing the game that we can move the knight in the 3+1 movement, in agreement with how other chess players move the same piece. So too in mathematics, for example, we might teach a pupil the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

However, in the absence of demonstrating how one applies (*uses*) the formula, could we reasonably claim that the pupil is a gifted 'language-user' in relation to the *use* of the quadratic formula? The formula does not guide *in and of itself*. A rule does not come equipped with a set of instructions as to its appropriate interpretation or use. Wittgenstein has made this much clear in the rule-following paradox, outlined earlier. The rule simply points – like a signpost (*Philosophical Investigations*, §85) – in the correct direction. Something more is needed.

Once again, the concept of *training* is central to how Wittgenstein resolves this conundrum:

> Following a rule is analogous to obeying an order. One is trained to do so, and one reacts in a particular way. But what if one person reacts to the order and training *thus*, and another *otherwise?* Who is right, then?
>
> (*Philosophical Investigations*, §206)

Wittgenstein concludes that this is resolved with reference only to the collection of demonstrated behaviours of like-minded language-users: 'Shared human behaviour is the system of reference by means of which we interpret an unknown language.' (Ibid.) Why, though, is the defining quality found in the agreement of a shared practice? Why did Wittgenstein not settle on the belief that the rule is enough to guide towards its appropriate use? Why is the rule – the signpost –not enough?

> Considered in itself a sign-post is just a board or something similar, perhaps bearing an inscription, on a post. Something so described does not, as such, sort behaviour into correct and incorrect – behaviour that counts as following the sign-post and behaviour that does not.
>
> (McDowell, 1992, p. 41)

An excellent example of this problem is found in Kripke (1982), in which there is a paradox which has famously become known as the 'Kripkenstein paradox'. An extension of Wittgenstein's original work on the peculiar nature of rule-following, Kripke poses an interesting conundrum in which the simplest of mathematical rules – addition – cannot be explained by the rule itself.

Kripke (1982, p. 9) poses a profound problem for mathematics teachers when he suggests that we might one day encounter a pupil who defines addition as follows:

$$x \bigoplus y = \begin{cases} x + y, & provided\ x, y < 57 \\ & 5\ otherwise \end{cases}$$

Kripke's problem presents us with a profound concern for teaching mathematics. Indeed, this function means that the pupil will add 'as normal' for all additions where the numbers being added are less than 57. This, Kripke suggests, may be because the list of examples shown to the pupil during teaching takes him naturally to this point (i.e. 57 is arbitrarily fixed). However, the serious concern arises post-teaching, when the pupil is asked to perform 68 + 57, and answers '5'. This new function, which Kripke calls 'quus' is the same as the rule for 'plus' up to all numbers just less than 57, but once we consider numbers 57 or more, all the answers become 5.

Kripke outlines why this is so problematic:

> Now, if the sceptic proposes his hypothesis sincerely, he is crazy; such a bizarre hypothesis as the proposal that I always meant quus is absolutely wild. Wild it indubitably is, no doubt it is false; but if it is false, there must be some fact about my past usage that can be cited to refute it. For although the hypothesis is wild, it does not seem to be *a priori* impossible.
>
> (Kripke, 1982, p. 9)

Alas, no such evidence can be provided. It seems as though the rule is followed 'blindly' and without good reason to lead towards the answer 125, rather than 5. The sceptic to whom Kripke refers is not suggesting that the answer to this problem *should* be 5; rather that there should be some reason why the answer is 125, and not 5. Kripke's efforts to appease the sceptic's concerns are not resolvable by looking *only* at the facts within the locality of the pupil. The rule for addition is no more a suitable candidate for the pupil continuing, than the quadition rule is. Addition and quadition are both signposts. In this case, they sit at the same crossroads: one leads towards an agreed way of acting (adding) and the other leads us on a strange and peculiar path (quadding). But when the pupil stands at the crossroads, equipped only with the rule and some finite list of example applications of the rule, Kripke concludes that there is no way to say what rule he should pick in order to go on *beyond the examples* he has been given. Rather worryingly, this does not just apply to a continuation of mathematics sequences and rules. As Bloor (1997, p. 10) opines, 'This does not just apply to number sequences. Teaching someone the word 'red' is, in a sense, teaching them the rule for using the word. This too involves moving from a finite number of examples to an open-ended, indefinitely large range of future applications.'

However, Kripke also considers why this finite list of examples do not explain the pupil's ability to follow the rule (addition) as we would want him to. Perhaps if we could see inside the pupil's mind, this would elicit some previously unforeseen information which would be able to tell us why he chooses 'plus' instead of 'quus'? Kripke outlines that this does not solve the problem, since 'whatever 'looking into my mind' may be, the sceptic asserts that even if God were to do it, he still could not determine that I meant addition by 'plus'.' (1982, p. 14), and that the sceptic 'claims that an omniscient being, with access to *all* available facts, still would not find any fact that differentiates between the plus and the quus hypothesis.' (1982, p. 39).

This is precisely because of the indeterminacy of a rule as a thing-in-itself, since 'instructions for following a rule underdetermine the correct way to follow a rule.' (Panjvani, 2008, p. 307). A finite list of examples, moreover, taken together with the rule we wish to teach, will not resolve this paradox, since 'finite behaviour cannot constrain itself to within uniqueness' (Wright, 1986, p. 98). Anscombe (1985, pp. 342-3) gives a resounding mathematical case to demonstrate this.:

> Although an intelligence tester may suppose that there is only one possible continuation of the sequence 2, 4, 6, 8, …, mathematical and philosophical sophisticates know that an indefinite number of rules are compatible with any such finite initial segment. So if the tester urges me to respond, after, 2, 4, 6, 8, …, with *the* unique appropriate next number, the proper response is that no such unique number exists.

Indeed, as Anscombe also outlines, the sequences whose nth term is defined by:

$$U_1(n) = 2n - \left(\frac{1}{24}\right)(n-1)(n-2)(n-3)(n-4)$$

$$U_2(n) = 2n + 45(n-1)(n-2)(n-3)(n-4)$$

$$U_3(n) = 2n - 3(n-1)(n-2)(n-3)(n-4)$$

will all generate the first 4 terms, 2, 4, 6, 8, … but all will produce radically different 5th terms. Perhaps, it might be contested, these rules – for quadding and for these over-complicated nth terms – are not the *simplest* rule; that is, they are not the *simplest* continuation of the finite list of examples given to us during teaching. Rather ominously, Chaitin (2007, p. 120-1) outlines – extending work on Turing's Halting Algorithm and Gödel's Incompleteness Theorems in his work on algorithmic information theory – that a simplest rule does not even exist.

So, what does this all point towards? Is the suggestion here, from Kripke, Wittgenstein and company that we cannot justify our pupils' ability to go beyond the information given to them in the finite list of examples shown to them in teaching? Quite the contrary, in fact: this points towards the realisation that there is some *other* aspect to learning which is fundamentally non-mental, non-internal, and non-rule-like, which allows the student to transcend the examples shown to him, and continue them *correctly.* Classrooms are not filled with children claiming to extend examples in varied and complicated ways. Teachers can feel comfortable in asking their students to 'go on in the same way…' after showing them a list of examples. Wittgenstein explains why:

> How can I follow a rule, when after all whatever I do can be interpreted as following it?
>
> It is true that *anything* can be somehow justified. But the phenomenon of language is based on regularity, on agreement in action.
>
> Here it is of the greatest importance that all of the enormous majority of us agree in certain things. I can, e.g., be quite sure that the colour of this object will be called 'green' by far the most of human beings who see it.
>
> … we say that, in order to communicate, people must agree with one another about the meaning of words. But the criterion for this agreement is not just agreement with reference to definitions, e.g., ostensive definitions – but *also* an agreement in judgements. It is essential for communication that we agree in a large number of judgements.
>
> (*Remarks on the Foundations of Mathematics,* VI, §§38-9)

Part of another proposition in Wittgenstein's consideration of this potentially troubling notion captures how we should think about this as educators. How should we teach? Wittgenstein says 'I will teach him … by *examples* and by *exercises* – And when I do this, I do not communicate less to him than I know myself.' (*Philosophical Investigations*, §208). Such teaching must take a very particular form: 'I do it, he does it after me; and I influence him by expressions of agreement, rejection, expectation, encouragement. I let him go his way, or hold him back; and so on.' (Ibid.). Going beyond the finite list of examples, applying the rules which are elicited within the examples, is the primary focus of our teaching anything: 'Teaching which is not meant to apply to anything but the examples given is different from that which *'points beyond'* them.' (Ibid.).

5 'Rails to Infinity'

Pointing beyond is crucial to following a list of finite examples and the rules which such examples encompass. The shared nature of human behaviour, which Wittgenstein calls a reference point, is what defines the concepts of correct and incorrect. Agreement is central to rule-following, and going beyond the information given to us within a finite segment of a sequence or a finite list of examples which are design to capture the meaning of the concept we are teaching. We might feel uncomfortable with this as an *explanation*, and we may want to search for some other reasons – usually internal reasons 'within' the learner – as to why our pupils generally choose the correct interpretation of our examples; that is the interpretation *we want* them to choose. However, as Kripke has shown, the reasons which we might offer to the sceptic who wishes to challenge us, when we look only within the locality of the learner – the within – soon give way to bedrock certainties.

Wittgenstein similarly warns:

> 'How am I able to follow a rule?' – If this is not a question about causes, then it is about the justification for my acting in *this* way in complying with the rule.
>
> Once I have exhausted the justifications, I have reached the bedrock, and my spade is turned. Then I am inclined to say: 'This is simply what I do.'
>
> (*Philosophical Investigations*, §217)

As unpleasant as it may seem to accept it, reasons come to an end, and we conclude that 'When I follow a rule, I do not choose. I follow the rule *blindly*.' (*Philosophical Investigations*, §219). Like 'rails invisibly laid to infinity' (§218) every rule has a meaning, captured in how it is used and applied by a similarly *trained* community of practitioners, and this rule 'traces the lines along which it is to be followed through the whole of space.' (§219). When we come to understand that the syntactic content of the rule (the formula, for example) is the *signpost*, and the semantic content of the rule – which is captured in how we are *shown* to *use* the rule in practice – are the *rails* *to infinity*, we come to appreciate how rule-following is central to how we learn anything, especially in mathematics. Namely, we refer to our specific *training* which has equipped us with the ability to interpret the rule correctly; a training which merged the *signpost* with the *rails to infinity*, to ensure that the 'infinitely long rails correspond to the unlimited application of a rule.' (§218).

Repeatedly *consistent application* of the rule *in agreement* with the adopted manner of its use is what points towards the rule being applied in the manner which is expected: ''But surely you can see … !' That's precisely the characteristic exclamation of someone who is compelled by a rule.' (*Philosophical Investigations*, §231). As Ryle (1949, p. 164) remarks:

> To settle whether a boy can do long division, we do not require him to try out his hand on a million, a thousand, or even a hundred different problems in long division. We should not be quite satisfied after one success, but we should not remain dissatisfied after twenty, provided they were judiciously variegated and that he had not done them before. A good teacher, who watched his procedure in reaching them, would be satisfied much sooner, and he would be satisfied sooner still if he got the boy to describe and justify the constituent operations that he performed.

Following a rule is therefore a compulsion to act, guided by the *training* which shows the syntactic rule, and the *carefully chosen examples* which lay down the rails to infinity. Oakeshott (1989, p. 50) summarises succinctly, thus:

> What is required in addition to information is knowledge which enables us to interpret it, to decide upon its relevance, to recognize what rule to apply and to discover what action permitted by the rule should, in the circumstances, be performed; knowledge (in short) capable of carrying us across those wide open spaces, to be found in every ability, where no rule runs.

6 On the Nature of Mathematics

Following the rules of mathematics is central to being able to *do* mathematics. The abstract constructs, algebraic norms and conventions, laws of arithmetic, geometry and trigonometry, and mathematical formulae are all syntactic, rule-like *signposts*, which point learners in the direction of what should be done. However, the *rails to infinity* are laid down by active engagement with the rule, by breathing interpretive understanding into the rule. Such understanding is found in the close imitation of a master expert, who has already seen the rule rolled out *ad infinitum*. The syntactic, informative rule is accompanied by the semantic element which is seen, and then practised. There are no further reasons given for trying to grasp *why* the rule is followed in the *correct* way. That is, we can appease Kripke's sceptic by saying that he was asking the right questions, but *looking in the wrong* *places* for his answer. When we consider *only the internal* aspects of the learner – what Kripke called 'looking into the mind' – we cannot find any reason to follow one interpretation of any rule over some other interpretation. We cannot, when restricting ourselves only to the inner, give logical grounds for claiming that 68 + 57 should be 125, rather than 5! Both solutions are viable, since both solutions are possible on *some interpretation of the rule*. What else is needed to justify the answer as 125? The *agreement* in *judgement* of the community of mathematicians to declare, unanimously, that '125' is the answer to 68 + 57. This much is clear from Wittgenstein's philosophy of rule-following and language-games, and it is significant for mathematics in particular: 'Indefinitely many other ways of acting are possible: but *we* do not call them 'following the rule'' (Malcolm, 1986, p. 155). Kripke's sceptic shows that there are infinitely many ways to follow the rule, but the mathematical community declare with one voice, which *one single way* is correct.

For this reason, Oakeshott suggests that 'before any skill or ability can appear, information must be partnered by 'judgement', 'knowing *how*' must be added to the 'knowing *what*' of information' (1989, p. 49). This notion should not trouble us too much, since there is an analogical extension of it within the philosophy of quantum physics, which often informs – due, mainly to the work of quantum pioneer Neils Bohr (1934, 1958, 1987) – the philosophy of psychological science. Similar concepts have been developed in the quantum movement, in relation to 'weak objectivity' (d'Espagnat, 1983), noting that pre-measurement states and post-measurement states are to be considered conceptually different, but logically connected through the principle known as 'complementarity', which Whitaker (1996, p. 184) and Katsumori (2011, pp. 18, 134) defines as 'mutual exclusion and joint completion'. Indeed, in a pre-measurement state – known as a 'superposition' in quantum physics – all things are possible. In their unmeasured state, Heisenberg's *Uncertainty Principle* guarantees that quantum objects have no specific location. They are both in Location A and Location B simultaneously. It is only once these entities are measured, they take on a *specific location*, either Location A *OR* Location B.

The lesson for Kripke's sceptic, and all other rule-governed behaviours, is simple: if we look only within the locality of the learner, and even if God were to look inside the local facts within him, He would still not be able to point towards the defining reason why 125 more reasonable than 5 as the answer to 68 + 57. Both answers are entirely in keeping with all the local facts about the child in their unmeasured state. However, the picture changes radically when the child is asked to publically declare their answer as '5' or '125', because this public declaration requires being *compared* to a pre-existing practice of mathematical engagements, in which all similarly trained mathematicians will *agree* that the answer to the question posed is '125' and not '5'. In this measured state, the child is now right *or* wrong; in their private moment, they were both right *and* wrong. Only when the situation becomes public is there are *reason* for saying one thing is correct, and the other not: 'If even God, who can see all the facts about the past (and into your mind), could not know that you meant addition then that doesn't illustrate limitations on God's knowledge. It shows that there is in this case *no fact for him to know.*' (Ahmed, 2007, p. 102, *emphasis added*). In essence: in our pre-measured, private moment, even God must conclude that '125' and '5' are *consistent* with some interpretation of the rule shown to us in our examples. Only when the matter is made public – and it steps into the domain of mathematical *agreements* and *judgements* of similarly trained people – is there a justification for appealing to the concepts of 'right' *OR* 'wrong'.

6 On How We Should Teach Mathematics

There are some general pedagogical points about how one should teach mathematics which arise from our considerations about rule-following and language-games. Wittgenstein's philosophy, rather than being behaviourist as some people may contend, is in fact fundamentally anthropological and sociological in nature; aspects which his philosophy shares with Bohr's philosophical ideas about the nature of physics, and indeed of psychology (Kitchen, 2017, pp. 191-205). These aspects of Wittgenstein's philosophy give rise to important corollary considerations about the nature of mathematics pedagogy.

Huge areas of mathematics are rule-like propositions, syntactic in nature. There is an abstractness about mathematics – something about which mathematicians like G.H. Hardy, for example were unapologetic – which makes it formal and logical. The principles of propositional calculus, for example, form a basis for analytical and logical thought within abstract mathematics. Their nature is overtly rule-like. Mathematics as a discipline rests on these types of precepts, and the foundations of mathematics, which Wittgenstein was particularly interested in, are fundamentally axiomatic. However, being able to recite these rules which form the basis of mathematical thought, in no way tells us how to use them; that is, how to be mathematicians. Wittgenstein remarks, 'Every sign *by itself* seems dead. *What* gives it life? – In use it *lives*. Is it there that it has living breath within it? – Or is the *use* its breath?' (*Philosophical Investigations*, §432).

The importance for mathematics pedagogy, therefore, is that the rules are in no way sufficient (or indeed necessary, for that matter) in order to become an accomplished mathematician. Teachers of mathematics should display the syntactic rules as signposts, and then breathe life into the sign by showing their pupils how the rule is to be used. We show this, as Wittgenstein suggests, in *examples* and *exercises* (*Philosophical Investigations,*

§208). Teachers of mathematics *do* mathematics in the presence of their students, and they model the prevailing practices of the community of similarly-trained mathematicians, to show their pupils how mathematics is done. In agreeing with their teacher, the pupil is agreeing with the practice of mathematics. We cannot castigate such agreement, nor suggest that learning mathematics can be done without it. If we remove the centrality of the teacher's representativeness of his practice, then we simultaneously remove the concept of correctness. It is 'we' who follow rules, and therefore, teaching mathematics should focus on developing an agreement in *how* the rules of mathematics are followed, and to making students part of the 'we' who follows them. Any efforts to obviate the need for rules and a public agreement with them, does not represent the nature of mathematics as a discipline. To appease Kripke's sceptic, for example, and to justify even the simplest forms of how mathematics is learnt (and to be able to say, assuredly, that 68 + 57 is 125!), we must accept that agreement with the mathematical tradition and prevailing mathematical practice is central to learning mathematical ideas.

The teacher of mathematics has a tremendously privileged position in this view of mathematical pedagogy. Indeed, it is they who represent the agreement which is shared among the 'we'. This agreement is made manifest in how they interact and discuss mathematics with students. If judgement is to be aligned with the rules and rule-like propositions of mathematics to enable students to *do* mathematics beyond the finite list of examples shown to them, then this judgement is tacitly imparted in the nuanced act of teaching. Almost anyone can impart a rule, but it takes a special character to impart the insight into how the rule is used. Through the teacher's gestures, his utterances, and his constant tussles to get his students to see what *he* sees, he imparts an insight into what he represents, at which point he brings his students into agreement with him, and *ipso facto*, the community of mathematicians from which he comes.

Wittgenstein's interlocutor poses a challenge: 'But do you really explain to the other person what you yourself understand? Don't you leave it to him to *guess* the essential thing? You give him examples – but he has to guess their drift, to guess your intention.' (*Philosophical Investigations,* §210). Wittgenstein rebuts, 'Every explanation which I can give myself I give to him too. 'He guesses what I mean' would amount to: 'various interpretations of my explanation come to his mind, and he picks one of them'. So in this case he could ask; and I could and would answer him.' (Ibid.) This important realisation outlines that 'Nothing is hidden' (§435). This captures the core elements of language-game, private language, and rule-following, to conclusively reveal that whatever can be thought, can also be said; so that whatever interpretation might take place in the private moment of the student, once revealed in the public realm it can be realigned, if necessary, on the rails to infinity, directed in the clear pathway outlined by the signpost rule.

Teaching mathematics, therefore, should be based on a rule-governed pedagogy, and the role of the teacher should be seen as the arbiter of correct and incorrect moves in the mathematical language-game. This teacher-centric model of teaching mathematics is unapologetic, since the teacher is the one who is equipped with the rules of mathematics and, indeed more importantly, the insight as to how these rules are used. Such insight – a public, community- and tradition-based concept, which encompasses the anthropological and sociological elements of mathematics as a discipline – is found by the student in closely following, and therefore eventually assimilating, the mannerisms and modes of acting of the master expert; the teacher.

Conclusions

Mathematics has a nuanced style, and its core precepts as a discipline are amenable to a very particular type of pedagogy. In this new educational age, when traditional methods of instruction are castigated form more inquiry-led and skills-based orthodoxies, it is significant to consider how mathematics, by its very nature as a discipline, may require a different pedagogy. Wittgenstein's philosophy is an important contributor to these considerations, not least because it can be argued that his work on rule-following and language-games seems to capture the essence of mathematics. Mathematics is both syntactic and semantic. It has rules (e.g. formulae and laws of logic), but these rules *in themselves* do not guide towards their appropriate use. They require life (meaning), which is breathed into them from the community of like-minded, similarly *trained* mathematicians. Teachers of mathematics are crucial in this process. They must serve diligently as the representative of their practice in the classrooms, and impart the norms and maxims of their study, which they were also shown in their own training. The teacher gives meaning to the rules which mathematics rests on since, as Wittgenstein argues, rule-following requires an active *agreement* in order to be said to be done in any meaningful manner. It is therefore central to mathematics teaching and learning, to recognise that in the absence of the 'we', which is the agreement with the practices, traditions, and customs of the mathematical community, no rule can be followed, and therefore mathematics – as a rule-governed practice – is rendered fundamentally inaccessible as a study.

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